1. Newton's Second Law of Motion.

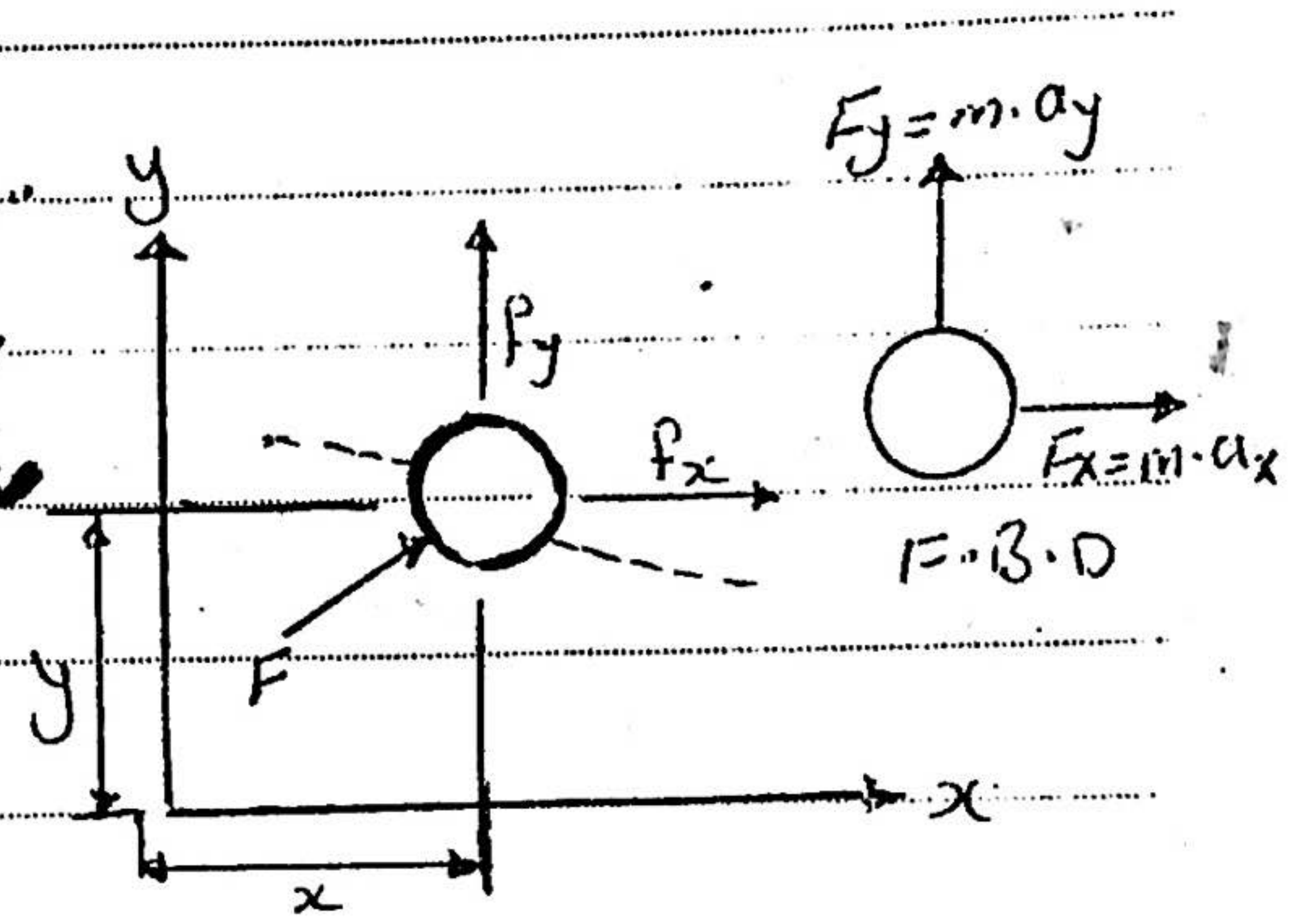
1.1 Motion of particles.

The general formula of Newton's scond law of motion Can be written as.

$$\Sigma F = m . a$$

which say that the resultant of all external forces acting on aparticle of mass (m) is must equal to multiply of the mass (m) and the acceleration of the particle (a).

* Rectilinear Coordinates.



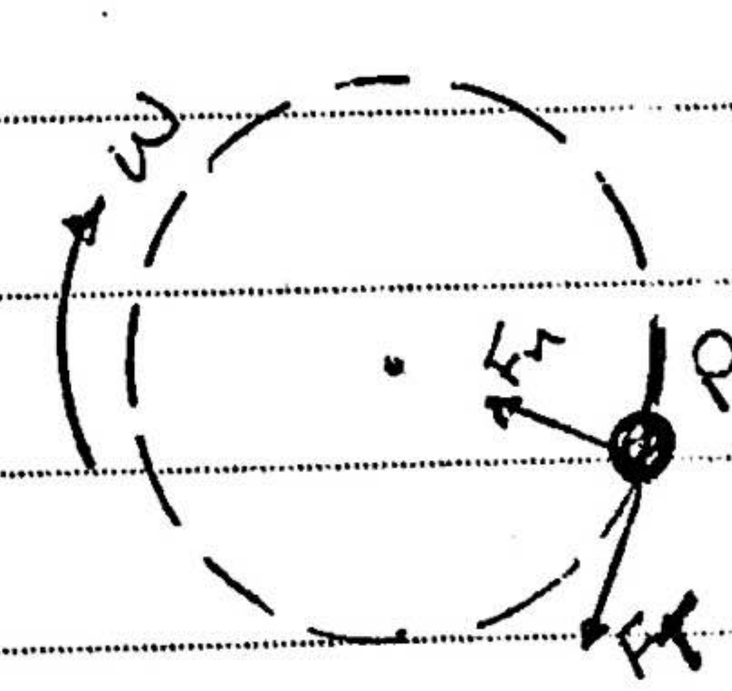
For this type of motion, the equation of motion Can be written as;

$$\Sigma F_x = m . a_x .$$

$$\Sigma F_y = m . a_y .$$

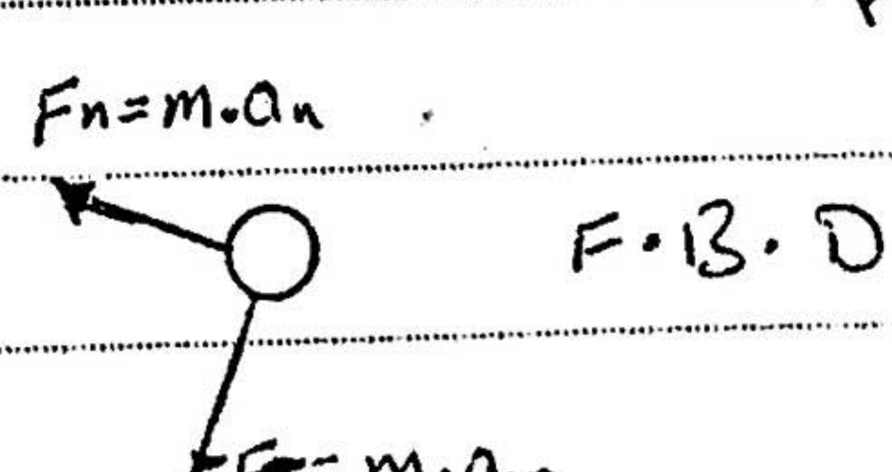
* Normal and Tangential Coordinates.

For this type of motion, the equation of motion (Newton's second law can be applied in the tangential and normal dire-ction), Can be written as;



$$\Sigma F_n = m . a_n .$$

$$\Sigma F_T = m . a_T .$$

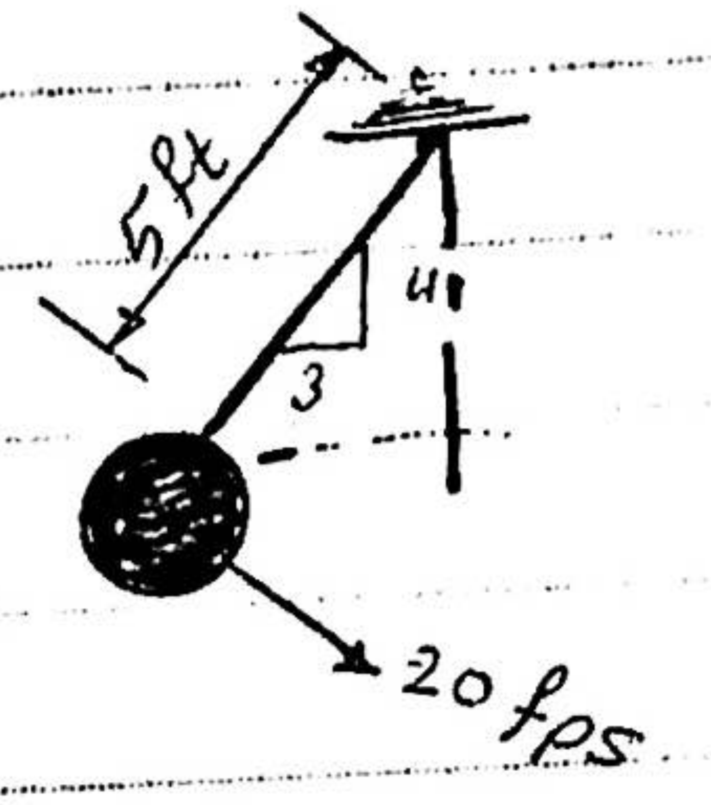


Example:

For the (8.05Ib) particle, determine

a. the tension (T) in the cord.

b. the angular acceleration of the particle.

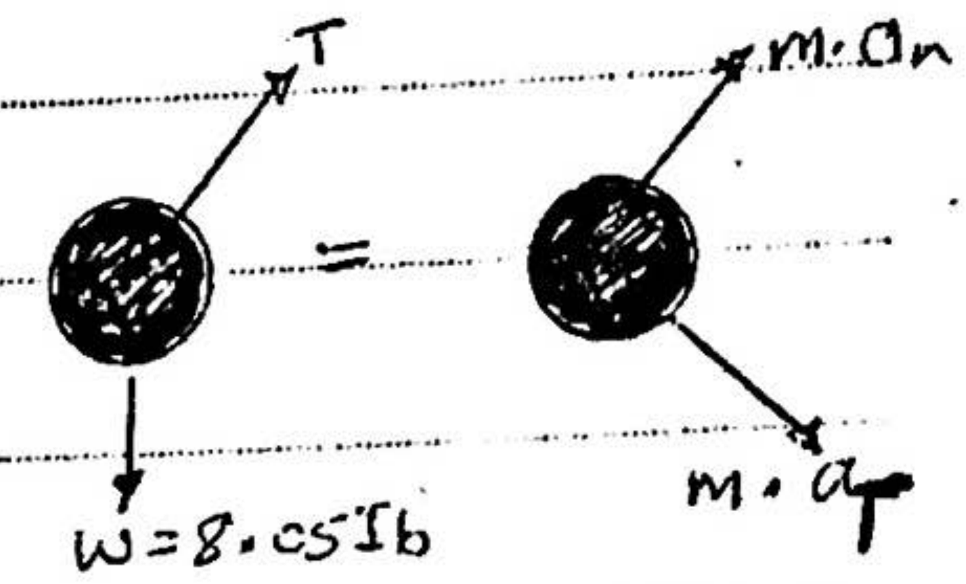


Solution:

$\omega = \frac{v}{r} = \frac{20}{5} = 4$ rad/sec

$a_n = \omega^2 \cdot r = (4)^2 \cdot (5) = 80$ fps^2 $\angle 53.13^\circ$

$a_T = \alpha \cdot r = 5\alpha$



$\Rightarrow \sum F_n = m \cdot a_n \Rightarrow T - W\cos 36.869^\circ = m \cdot a_n$

$T - (8.05)(\cos 36.869^\circ) = (8.05/32.2) * 80$

$\therefore T = 26.44$ Ib.

$\Rightarrow \sum F_T = m \cdot a_T \Rightarrow 8.05 \sin 36.869^\circ = (8.05/32.2)\, a_T$

$\therefore a_T = 19.32$ fps^2

$\Rightarrow \therefore a_T = \alpha \cdot r$

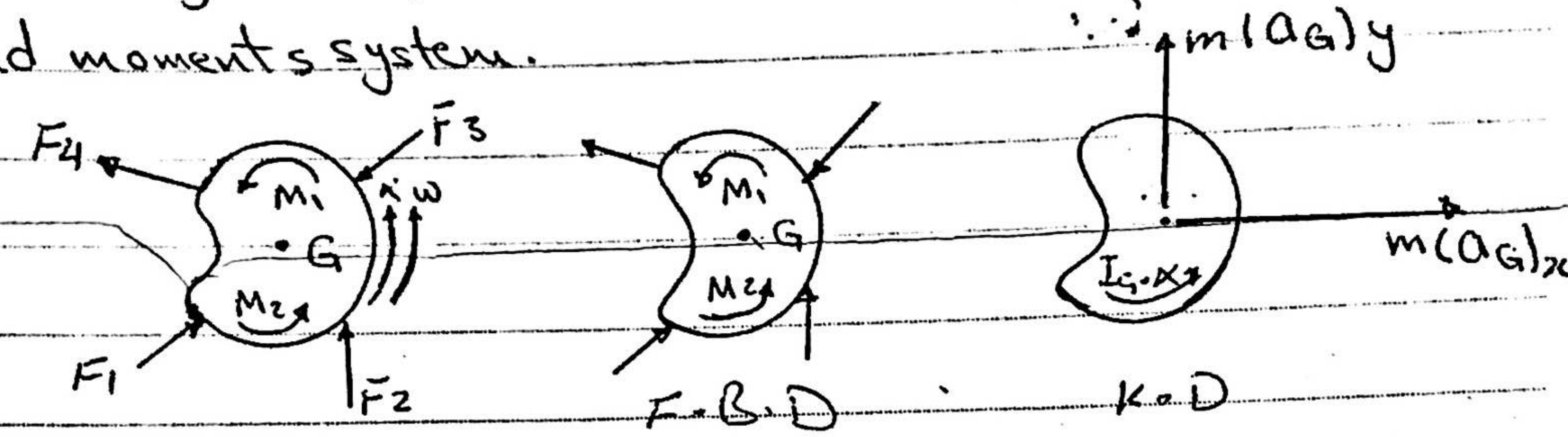
$19.32 = \alpha * (5)$

$\Rightarrow \alpha = 19.32/5$

$= 3.864$ rad/sec^2

1.2 General plane Motion of a Rigid Body.

Consider the rigid body in the figure, which is subjected to general plane motion caused by the external forces and moments system.



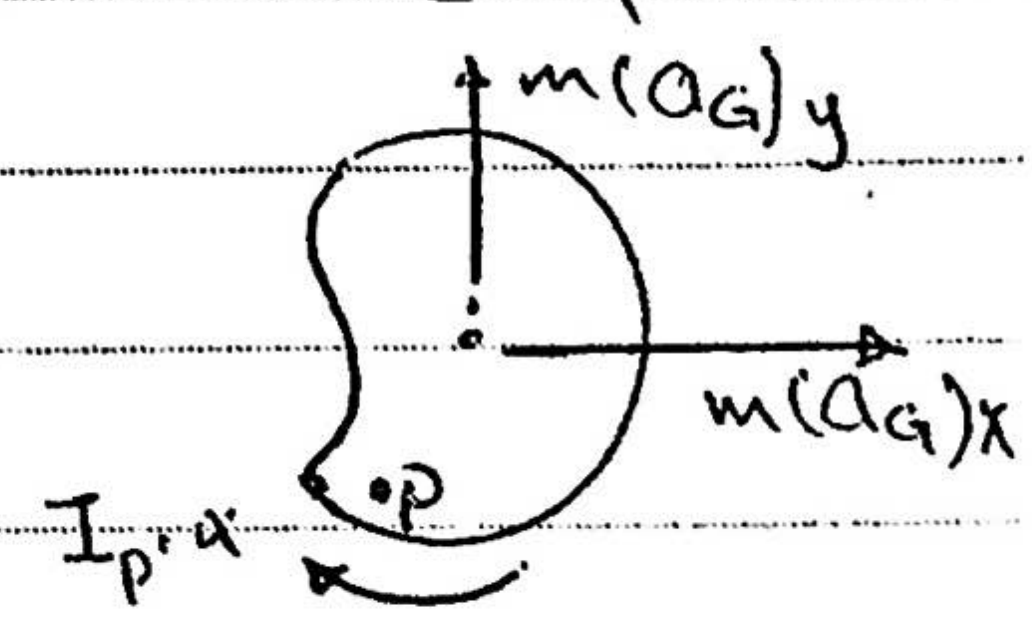
With considering the x and y coordinates system, the three equations of motion becomes;

$$\sum F_x = m.(a_G)_x .$$

$$\sum F_y = m.(a_G)_y .$$

$$\sum M_G = I_G . \alpha .$$

In general, it may be convenient to sum moments about some point (p) other than (G). Then the equations of motion become;



$$\sum F_x = m(a_G)_x .$$

$$\sum F_y = m(a_G)_y .$$

$$\sum M_p = \sum (M_k)_p .$$

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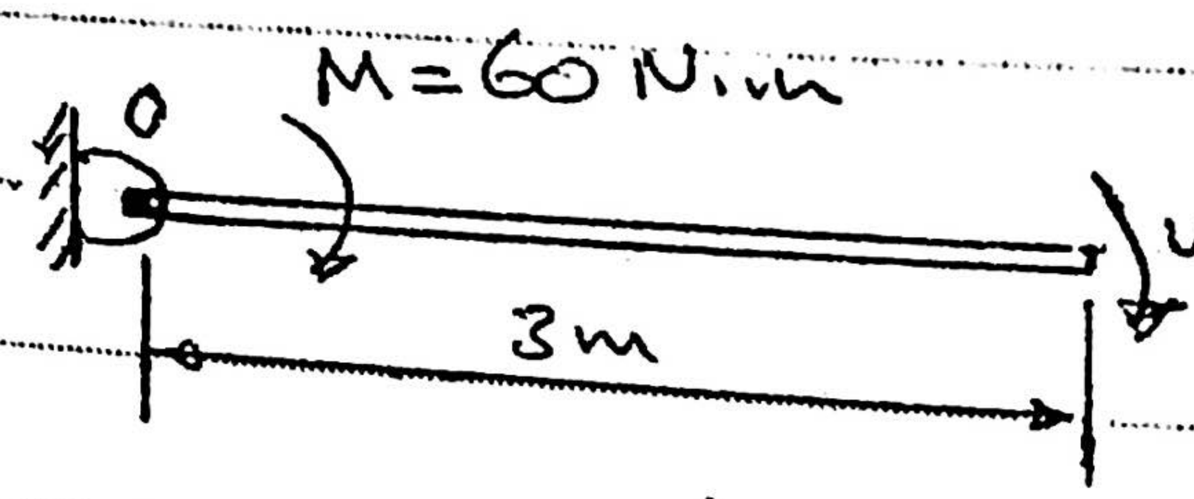
Example:

The 20 kg slender rod shown in the figure, is rotating in the vertical plane, and at the instant shown it has an angular velocity of $\omega = 5$ rad/sec. Determine the rod's angular acceleration and the horizontal and vertical components of reaction at the pin at this instant.

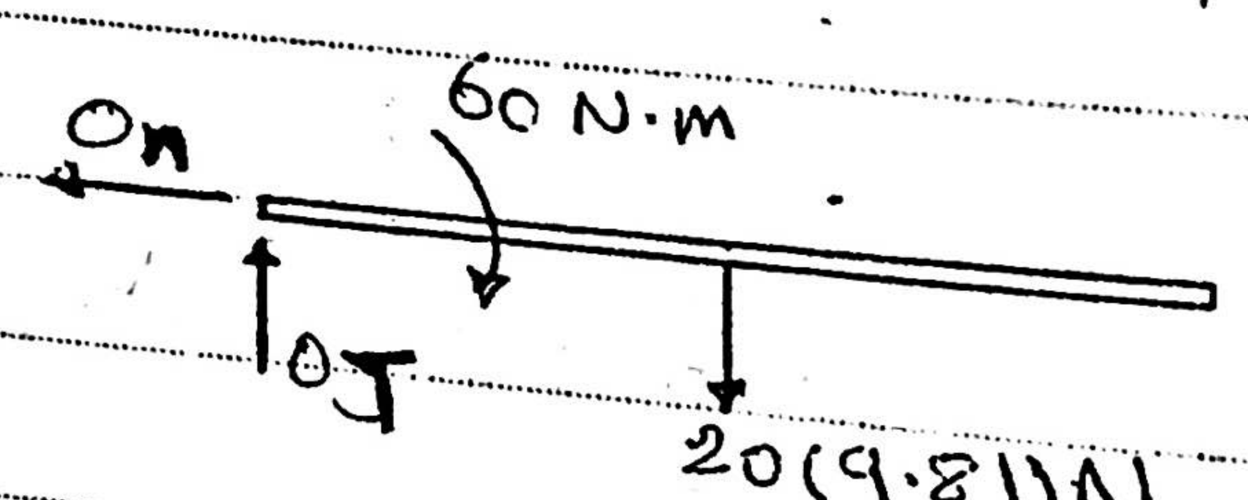
Solution:



$\overset{+}{\leftarrow} \sum F_n = m . a_n$

$= m . \omega^2 . r_G$

$\Rightarrow O_n = (20)(5)^2 . (1.5)$



Force

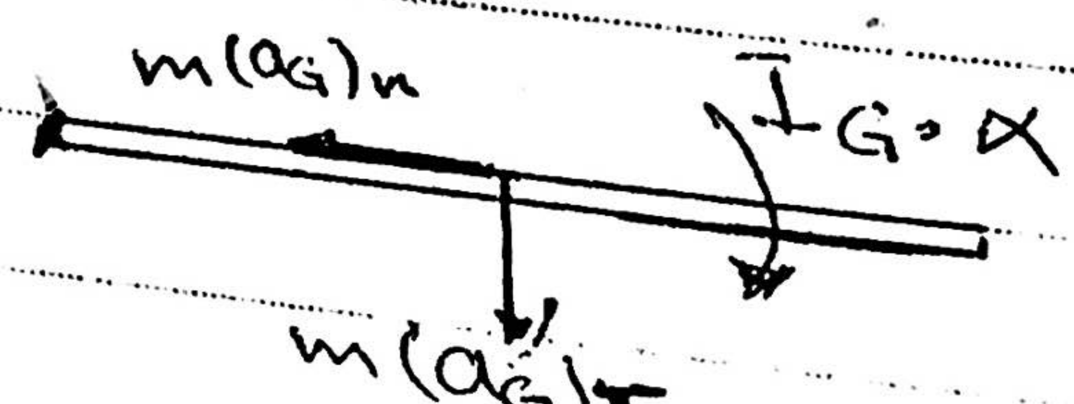
$+\downarrow \sum F_T = m . a_T$

$= m . \alpha . r_G$

$\Rightarrow -O_T + 20(9.81) = (20)(\alpha)(1.5)$

$\circlearrowright \sum M_G = I_G . \alpha$

$\Rightarrow O_T(1.5) + 60 = \frac{1}{12}(20)(3)^2 . \alpha$



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by solving $O_n = 750N$, $O_T = 19N$, and $\alpha = 5.9$ rad/s^2

OR

$\circlearrowright \sum M_O = \sum (M_k)_O$

$\Rightarrow 60 + (20)(9.81)(1.5) = \frac{1}{12}(20)(3)^2 \alpha + (20)(1.5)(1.5)\alpha$

$\Rightarrow \alpha = 5.9$ rad/s^2

$= \frac{mL^2}{3} * \alpha$

2. Second-Order Ordinary Differential Equations.

2.1 Homogenous Equation ($f(t)=0$)

Take the equation;

$$\frac{d^2x}{dt^2}+a_1\cdot\frac{dx}{dt}+a_2x=0$$

let $(d/dt)=S$;

$\Rightarrow$ the above equation become

$S^2x+a_1Sx+a_2x=0$.

$\therefore\ S^2+a_1S+a_2=0$

$\therefore\ S_{1,2}=\dfrac{-(a_1)\mp\sqrt{(a_1)^2-4a_2}}{2}$

Now, * If $S_1=S_2=S$ (Real and equal roots)

$\Rightarrow$ the solution is $x(t)=(A+Bt)e^{st}$

* If $S_1\neq S_2$ (Real and different roots)

$\Rightarrow$ the solution is $x(t)=A.e^{s_1t}+B.e^{s_2t}$

* If S_1 and S_2 are an imaginary roots,

$S_1=\alpha$ and $S_2=\beta$

$\Rightarrow$ the solution is $x(t)=e^{\alpha t}(A\sin\beta t+B\cos\beta t)$.

But if the equation in the form;

* $\dfrac{d^2x}{dt^2}+n^2x=0\Rightarrow$ the solution is $x(t)=A\cos nt+B\sin nt$

* $\dfrac{d^2x}{dt^2}-n^2x=0\Rightarrow$ the solution is $x(t)=A\cosh nt+B\sinh nt$.

Example: Solve $4\dfrac{d^2x}{dt^2}-8\dfrac{dx}{dt}+7x=0$

Solution: $4s^2-8s+7=0\Rightarrow S_{1,2}=\dfrac{-(-8)\mp\sqrt{(-8)^2-(4)(4)(7)}}{(2)(4)}$

$\Rightarrow S_{1,2}=1\mp i(\sqrt{3}/2)\Rightarrow\alpha=1,\ \beta=\sqrt{3}/2$

$\Rightarrow x(t)=e^{\alpha t}(A\cos\beta t+B\sin\beta t)=e^{t}(A\cos\frac{\sqrt{3}}{2}+B\sin\frac{\sqrt{3}}{2})$

* Syllabus:

CH. 1: Oscillatory Motion.

CH. 2: Free Vibration.

CH. 3: Harmonically – Excited Vibration.

CH. 4: Transient Vibration.

CH. 5: Two Degree Of Freedom.

CH. 6: Multi Degree Of Freedom.

* References:

1. Theory of Vibration With Application.
   By: W. T . Thomson . 1972.
2. Theory of Vibration With Application. Third Edition.
   By: W. T . Thomson . 1988.
3. Mechanical Vibrations. Third Edition.
   By: RAO . 1995.

Introduction:

There are two general classes of vibrations - free and forced. ***Free vibration*** takes place when a system oscillates under the action of forces inherent in the system itself, and when external impressed forces are absent. The system under free vibration will vibrate at one or more of its ***natural frequencies***, which are properties of the dynamic system established by its mass and stiffness distribution.

Vibration that takes place under the excitation of external forces is called ***forced vibration***. When the excitation is oscillatory, the system is forced to vibrate at the excitation frequency. If the frequency of excitation coincides with one of the natural frequencies of the system, a condition of ***resonance*** is encountered, and dangerously large oscillations may result. The failure of major structures such as bridges, buildings, or airplane wings is an awesome possibility under resonance. Thus, the calculation of the natural frequencies of major importance in the study of vibrations.

Vibrating systems are all subject to ***damping*** to some degree because energy is dissipated by friction and other resistances. If the damping is small, it has very little influence on the natural frequencies of the system, and hence the calculation for the natural frequencies are generally made on the basis of no damping. On the other hand, damping is of great importance in limiting the amplitude of oscillation at resonance.

The number of independent coordinates required to describe the motion of a system is called ***degrees of freedom*** of the system. Thus, a free particle undergoing general motion in space will have three degrees of freedom, and a rigid body will have six degrees of freedom, i.e., three components of position and three angles defining its orientation. Furthermore, a continuous elastic body will require an infinite number of coordinates (three for each point on the body) to describe its motion; hence, its degrees of freedom must be infinite. However, in many cases, parts of such bodies may be assumed to be rigid, and the system may be considered to be dynamically equivalent to one having finite degrees of freedom. In fact, a surprisingly large number of vibration problems can be treated with sufficient accuracy by reducing the system to one having a few degrees of freedom.

Chapter One

Harmonic Motion

Oscillatory motion may repeat itself regularly, as in the balance wheel of a watch, or display considerable irregularity, as in earthquakes. When the motion is repeated in equal intervals of time *T*, it is called ***period motion***. The repetition time t is called the ***period*** of the oscillation, and its reciprocal, $f = 1/\tau$, is called the ***frequency***. If the motion is designated by the time function x(t), then any periodic motion must satisfy the relationship

$$x(t) = x(t + \tau).$$

Harmonic motion is often represented as the projection on a straight line of a point that is moving on a circle at constant speed, as shown in Fig. 1. With the angular speed of the line o-p designated by w , the displacement x can be written as

$$x = A \sin \omega t \qquad \text{......(1)}$$

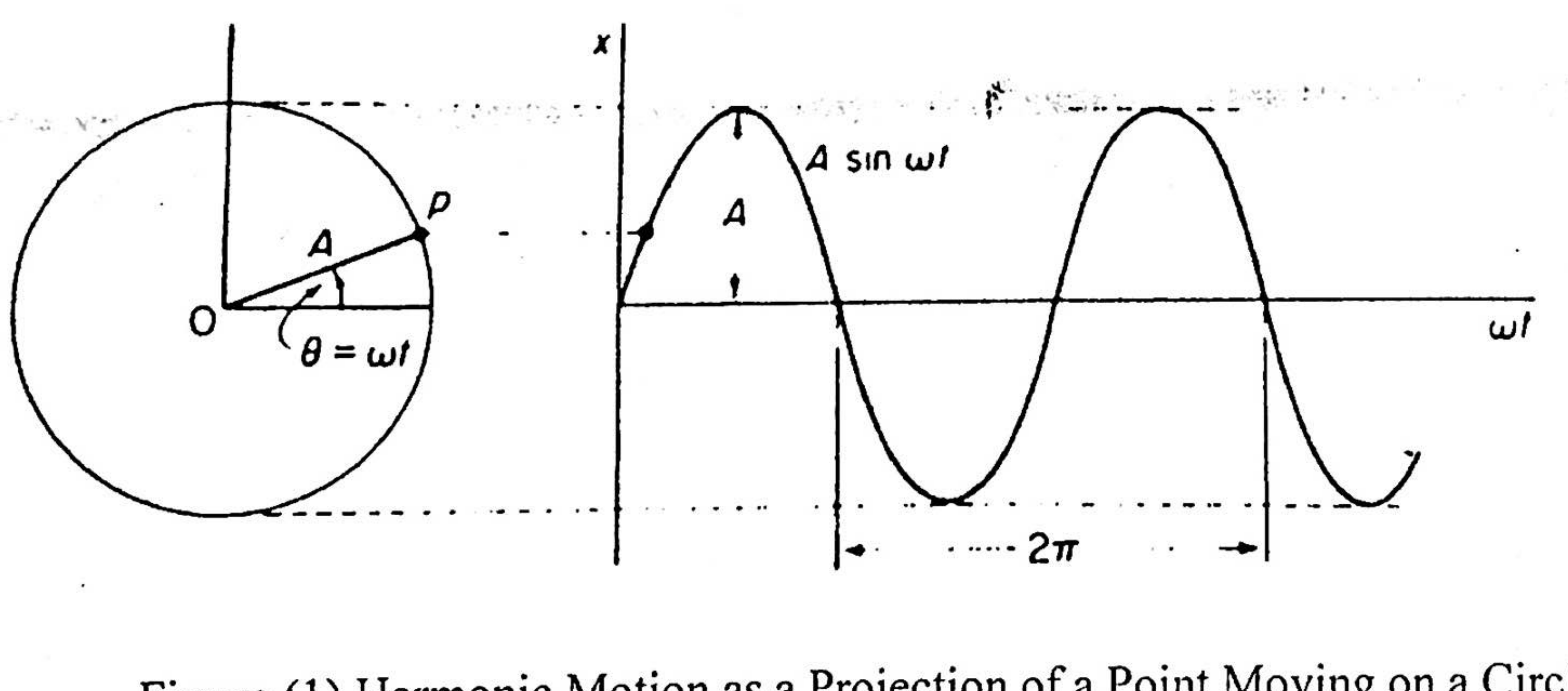


Figure (1) Harmonic Motion as a Projection of a Point Moving on a Circle

The quantity w is generally measured in radians per second, and is referred to as the ***angular frequency***. Because the motion repeats itself in 2p radians, we have the relationship

$$\omega = \frac{2\pi}{\tau} = 2\pi f \qquad \text{......(2)}$$

where t and f are the period and frequency of the harmonic motion, usually measured in seconds and cycles per second, respectively.

The velocity and acceleration of harmonic motion can be simply determined by differentiation of Eq. 1.

Using the dot notation for the derivative, we obtain.

$$\dot{x}=\omega A\cos\omega t=\omega A\sin(\omega t+\pi/2) \qquad \text{......(3)}$$

$$\ddot{x}=-\omega^2 A\sin\omega t=\omega^2 A\sin(\omega t+\pi) \qquad \text{......(4)}$$

* The angular frequancey (natural) :-

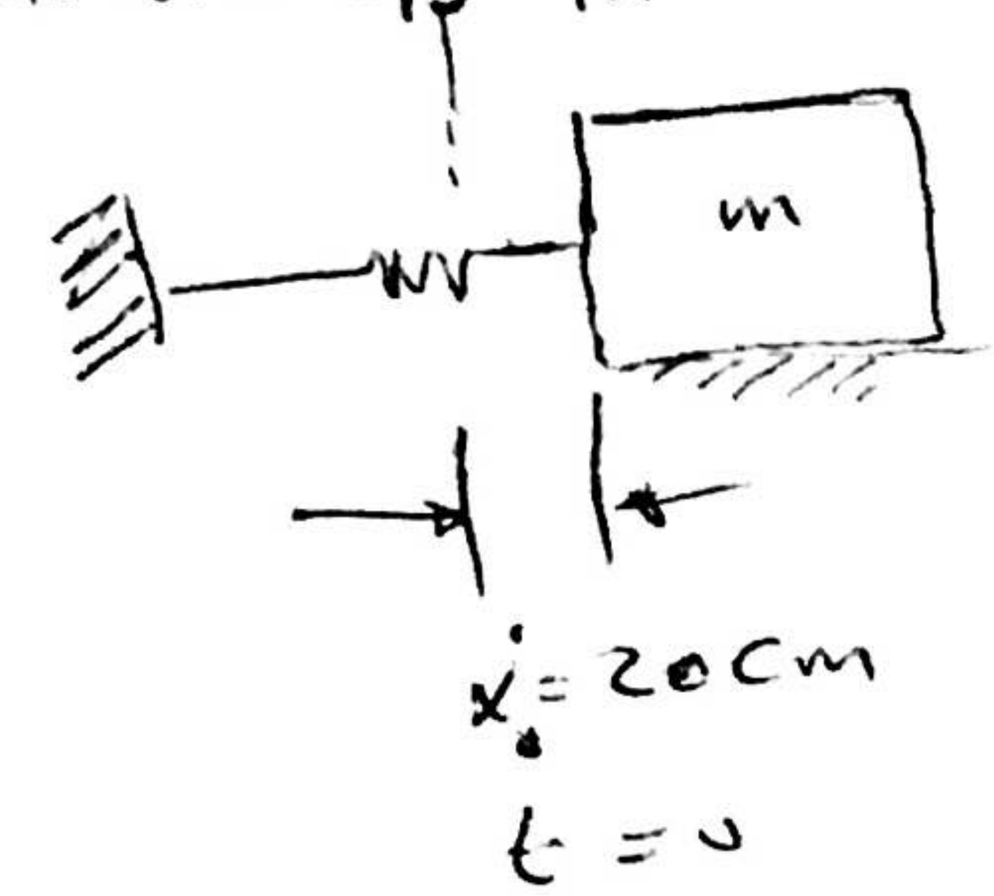
$$\omega_n=\sqrt{\frac{K}{m}}$$

K :- Constant of stiffness (N/m)
m :- mass of body (kg)

Example :- The system show in Fig. It is attached to spring pulled 20 cm to the right and released at t = 0. The total oscillations 15 in 10. sec

Determine :-

1. The Period of oscillation
2. Natural frequency
3. Max speed



① $\tau = ?$ $\quad \tau = \dfrac{\text{total time}}{\text{total oscillition}} = \dfrac{10}{15} = \dfrac{2}{3}$ sec

② $f = \dfrac{1}{\tau} = \dfrac{1}{2/3} = \dfrac{3}{2} = 1.5\ s^{-1}$ or Hz

③ $x(t) = A \cdot \sin(\omega t + \phi)$

at $t = 0$ $\quad x_0 = 20 * 10^{-2}$ m $\quad$ sub in equation

$x(0) = A \cdot \sin\phi \Rightarrow 20 * 10^{-2} = A \cdot \sin\phi$ — ①

$\dot{x}(t) = A \cdot \omega \cdot \cos(\omega t + \phi)$ — 2

at $t = 0$ $\quad \dot{x} = 0$

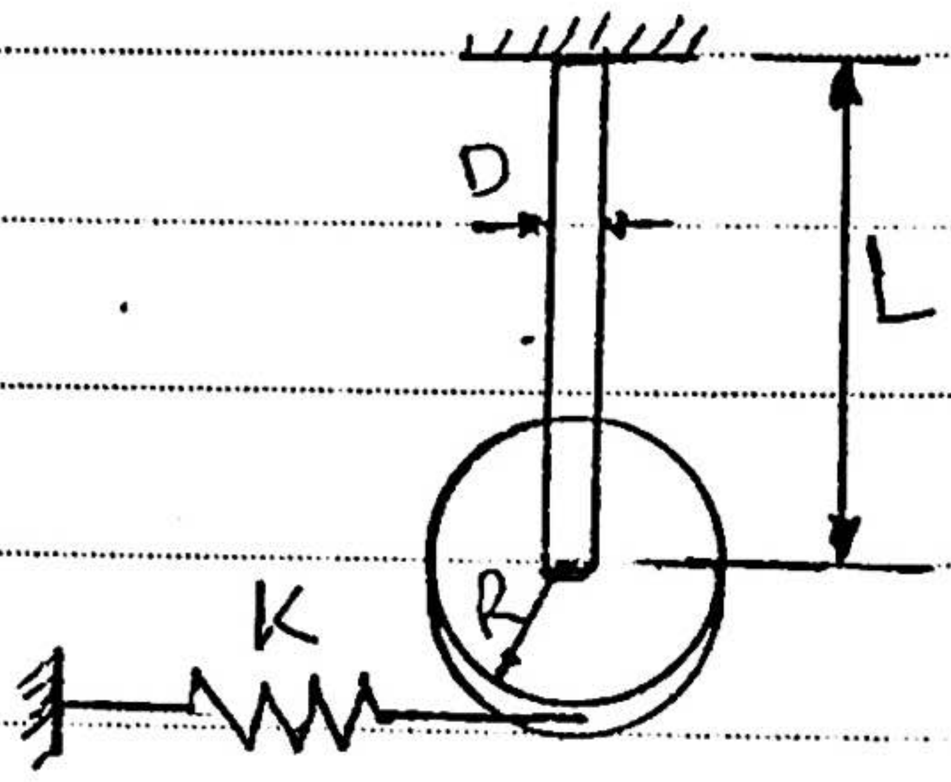
$0 = A \cdot \omega \cdot \cos\phi \Rightarrow A\omega \neq 0 \therefore \cos\phi = 0 \Rightarrow \phi = \dfrac{\pi}{2}$

$\therefore 20 * 10^{-2} = A * 1 \Rightarrow A = 20 * 10^{-2}$ m $\quad$ sub in ②

$\dot{x}(t) = 20 * 10^{-2} * \dfrac{2\pi}{2/3} = 1.88$ m/sec.

Example: For the torsional system shown in figure, the weight of the disc is (193.2 Ib), (L = 31.4 in) (D = 2 in) and (R = 4 in). If the natural frequency is (87.2 Hz), determine the shear factor of the shaft. K = 20 lb/in

Solution:



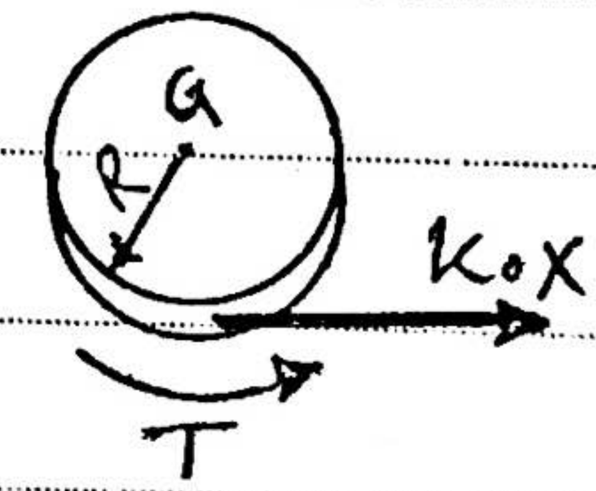
* For the shaft.

$$\frac{T}{J} = \frac{G.\theta}{L} \Rightarrow T = \frac{G.J}{L}.\theta$$

$$\therefore K_T = \frac{G.J}{L} \quad \text{and} \quad W_n = \frac{2\pi}{\tau} = 2\pi f$$

$$\therefore W_n = 547.894$$

* For the disc.



$$\circlearrowleft + \sum M_G = I\ddot{\theta}$$

$$\Rightarrow -T - K.x.R = I\ddot{\theta} \quad , \text{But } x = R\theta$$

$$-K_T\theta - KR^2\theta = I\ddot{\theta}$$

$$\ddot{\theta} + \left(\frac{K_T + KR^2}{I}\right)\theta = 0$$

$$\therefore W_n = \left(\frac{K_T + KR^2}{I}\right)^{1/2} = \left(\frac{G.J + K.R^2.L}{L.I}\right)^{1/2} = 547.894 \text{ rad/s}$$

assume $k = 20$ Ib/in

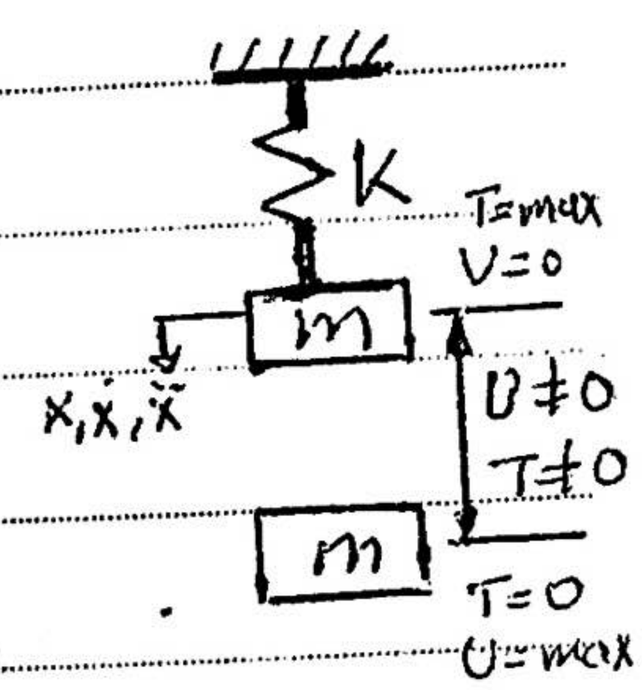
$$I = \frac{1}{2} mR^2 = \frac{1}{2}\left(\frac{193.2}{32.2 * 12}\right)(4)^2 = 4 \text{ Ibm.in}^2$$

$$J = \frac{1}{2}\pi r^4 = \frac{1}{2}.\pi.(1)^4 = \pi/2 \quad \text{or} \quad J = \frac{\pi}{32} D^4$$

All above data be substited in the angular frequency equation to get; $G = 2.399 * 10^7$ Ib/in²

Energy Method:

For the free vibration of undamped system, the energy is partly kinetic and partly potential. In the conservative system the total energy (kinetic + potential) is constant and its rate of change is zero, so the energy method can be used to derive the equation of motion as follows:



$$T + U = \text{Constant}$$

$$T_1 + U_1 = T_2 + U_2$$

and $\frac{d}{dt}(T + U) = 0$

where: T: is the kinetic energy and (U) is potential energy

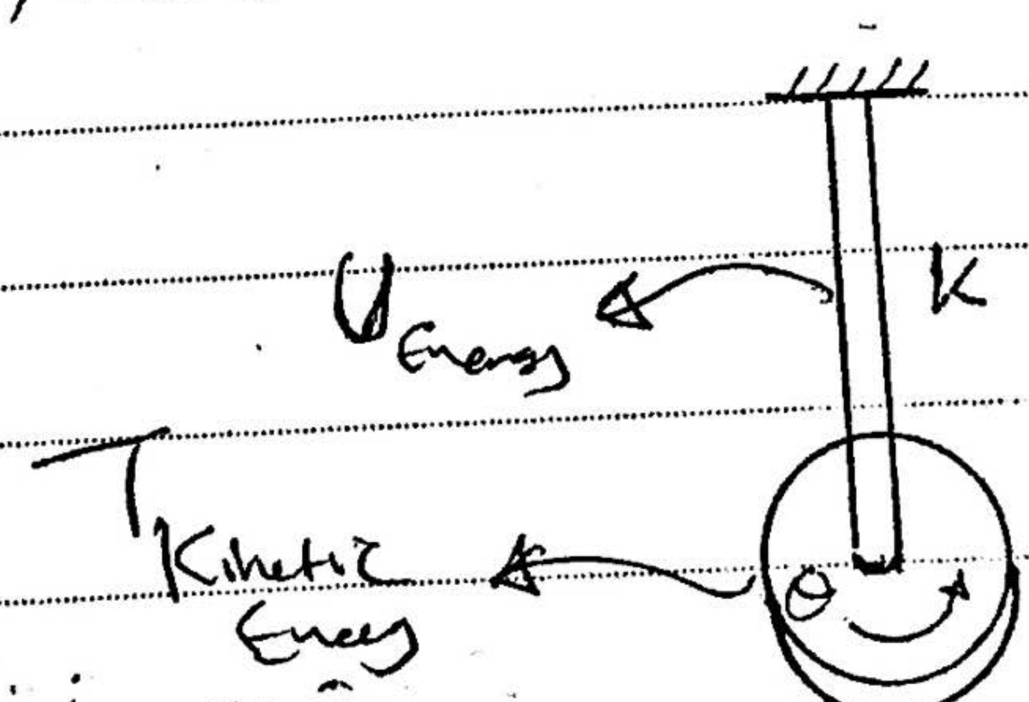
* If the system is undergoing harmonic motion, then

$$T_{max} = U_{max}$$

Example:

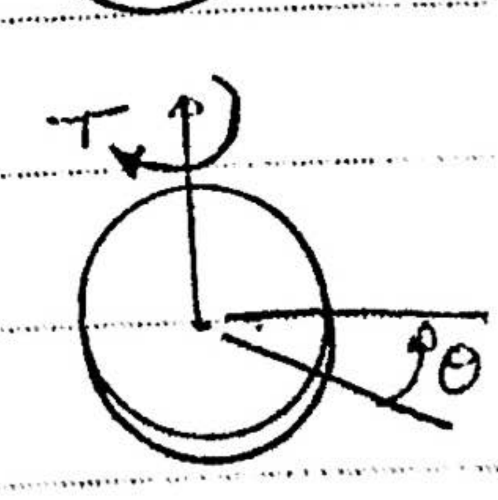
Determine the natural frequency of the torsional pendulum where (k) torsional stiffness $\left(\frac{N \cdot m}{rad}\right)$ and (I) mass moment of inertia $(N.m.s^2)$.

Solution:



1. By Energy Method

$$T = \frac{1}{2} I . \dot{\theta}^2$$



$$\therefore U = \frac{1}{2} K . \theta^2$$

$\Rightarrow$ Continue

$T_{max} = U_{max}$

$T_{max} = \frac{1}{2} I \dot{\theta}_{max}^2 = \frac{1}{2} I \omega_n^2 \theta^2$

$U_{max} = \frac{1}{2} k \theta_{max}^2 = \frac{1}{2} k \theta^2$

$\Rightarrow \frac{1}{2} I \omega_n^2 \theta^2 = \frac{1}{2} k \theta^2 \Rightarrow \omega_n^2 = \frac{k}{I}$

[illegible]

$T + U = \text{Constant}$

$\frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} k \theta^2 = C \quad] * 2$

$I \dot{\theta}^2 + k \theta^2 = 2C$

differentiating w.r.t. time.

$I \cdot 2 \dot{\theta} \ddot{\theta} + k \cdot 2 \theta \dot{\theta} = 0$

$I \ddot{\theta} + k \theta = 0$

$\ddot{\theta} + \frac{k}{I} \theta = 0 \Rightarrow \omega_n^2 = \frac{k}{I}$

2. By simple torsion theory

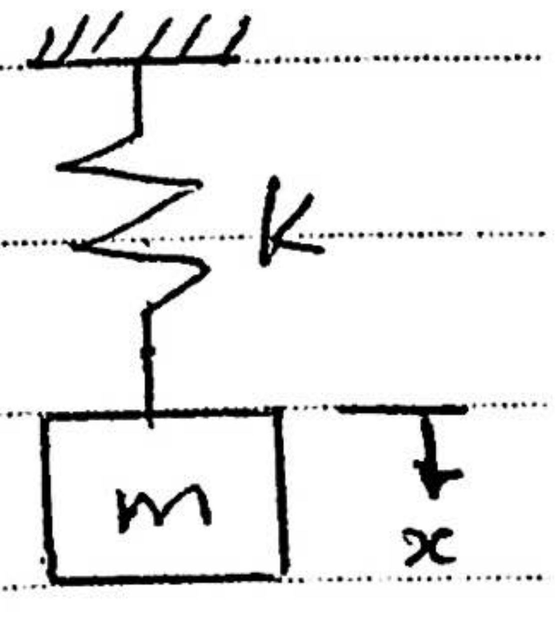
$\sum M = I \ddot{\theta}$

$\Rightarrow \quad I \ddot{\theta} = -k \theta$

$I \ddot{\theta} + k \theta = 0 \Rightarrow \ddot{\theta} + \frac{k}{I} \theta = 0 \Rightarrow \omega_n^2 = \frac{k}{I}$

Example:

Find the natural frequency and equation of motion for the system as shown.



Solution:

$T + U = C$

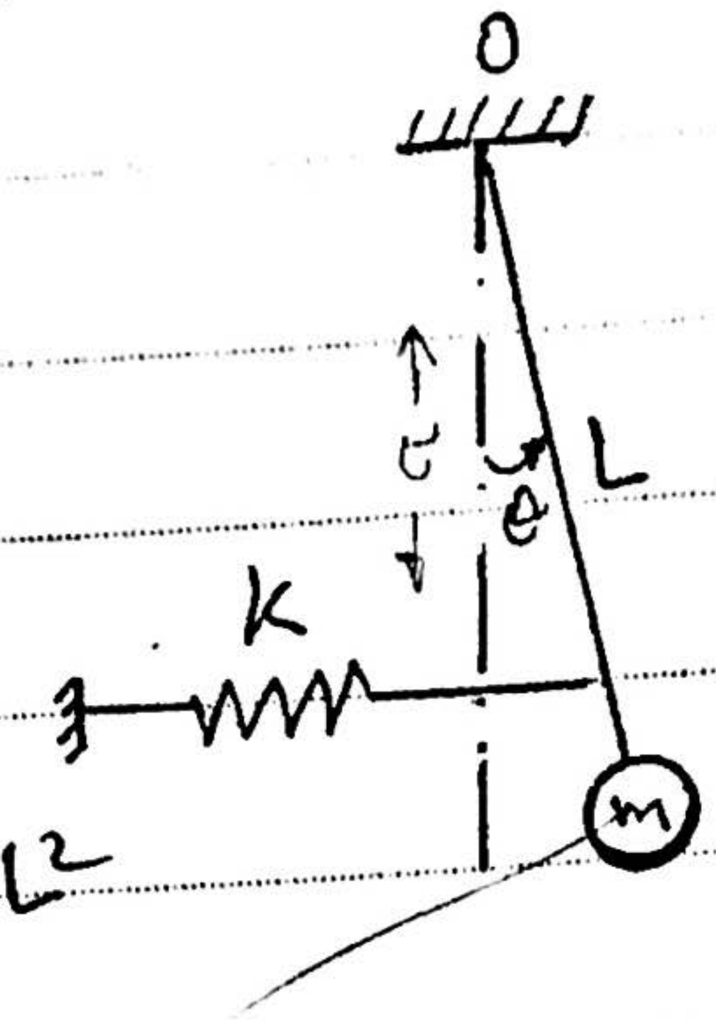
$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = C$

By drivitive

$2 * \frac{1}{2} \cdot m \cdot \dot{x} \cdot \ddot{x} + \frac{1}{2} * 2 \cdot k \cdot x \cdot \dot{x} = 0$

$m \ddot{x} + k x = 0 \Rightarrow \ddot{x} + \frac{k}{m} x = 0 \Rightarrow \omega_n^2 = \frac{k}{m}$

(3) The mass of the rod as a shown is small compared to the mass attached to it. For small oscillation, calculate the natural frequency of the swing of the mass. Assume $I_o = m.L^2$



Solution:

$\sum M_o = I\ddot{\theta}$

$\Rightarrow -mg\sin\theta . L - K . X . a = I\ddot{\theta}$

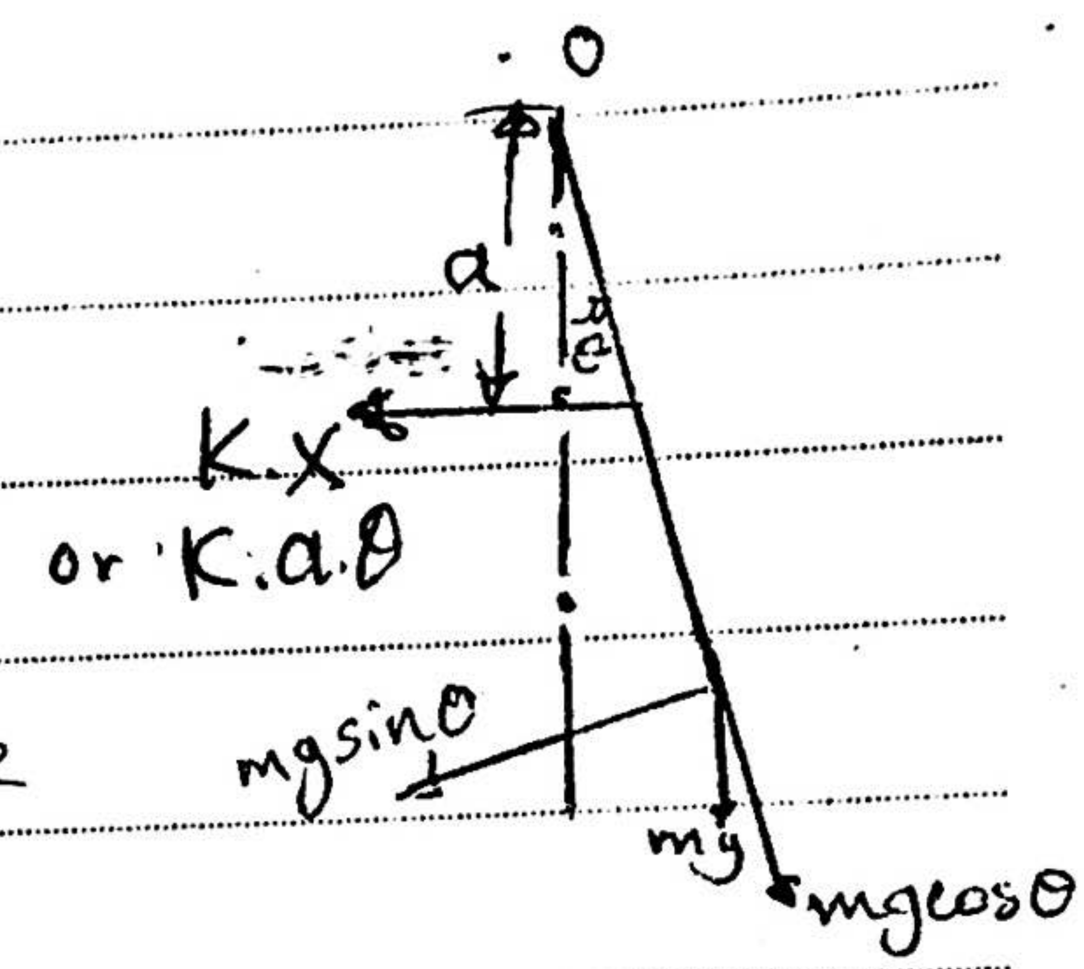
But $\sin\theta = \theta$; (small oscillation)

$-mg.\theta . L - K a \theta . a = m . L^2$

$\Rightarrow mL^2\ddot{\theta} + (mgL + ka^2)\theta = 0$

$\therefore \omega_n = \left(\dfrac{mgL + ka^2}{mL^2}\right)^{1/2} \Rightarrow f_n = \dfrac{\omega_n}{2\pi}$

* By Energy Method:



$T = \frac{1}{2} I\dot{\theta}^2$ (K.E of mass)

$U = U_{weight} + U_{spring}$

$= m.g.h + \frac{1}{2} k x^2$

$= mg(L - L\cos\theta) + \frac{1}{2} k (a\theta)^2$

$\therefore \dfrac{d}{dt}(T + U) = 0$

$\Rightarrow \frac{1}{2} I . 2\dot{\theta}\ddot{\theta} + mg(0 - L(-\sin\theta)\dot{\theta}) + \frac{1}{2} k a^2 . 2\theta\dot{\theta} = 0$

$\Rightarrow mL^2\ddot{\theta} + mgL\theta + ka^2\theta = 0$

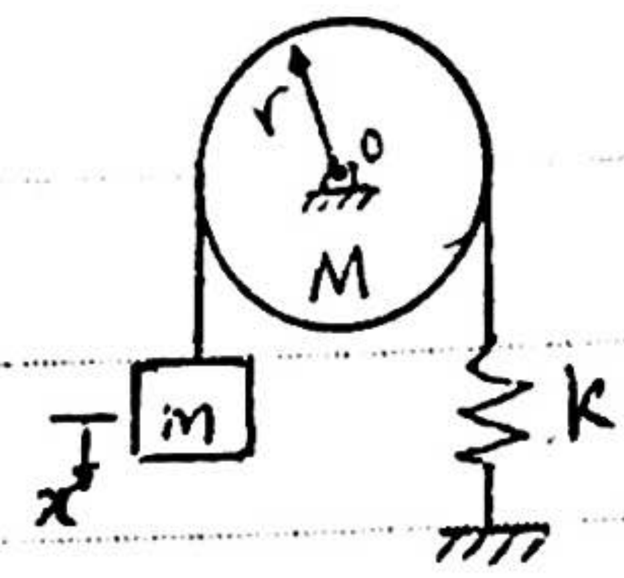
$\therefore \ddot{\theta} + \dfrac{mgL + ka^2}{mL^2}\theta = 0$

$\omega_n = \left(\dfrac{mgL + ka^2}{mL^2}\right)^{1/2}$

$f_n = \dfrac{\omega_n}{2\pi} = \dfrac{1}{2\pi}\left(\dfrac{mgL + ka^2}{mL^2}\right)^{1/2}$

(4) Determine the natural frequency of the spring-mass-pulley as shown below.

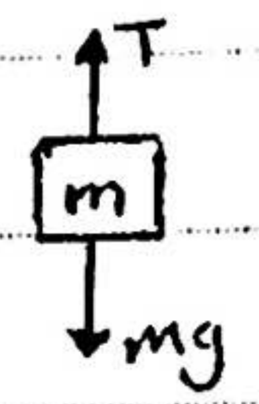
Solution:



* For mass (m).

$\sum F_y = m\ddot{x} \Rightarrow -T = m\ddot{x}$ ....(1)

* For pulley (M).

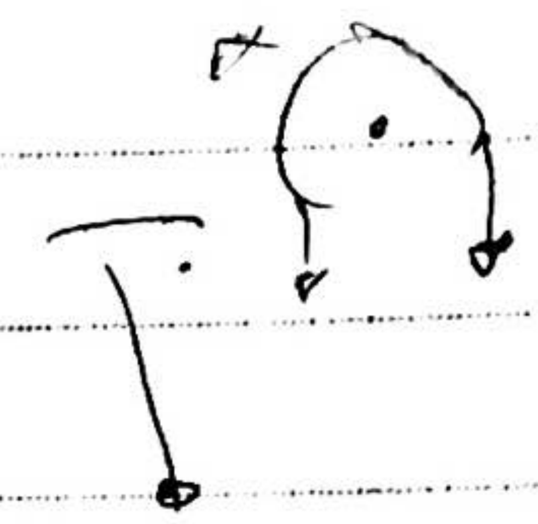


$\sum M_o = I\ddot{\theta} \Rightarrow T.r - k.r^2.\theta = I\ddot{\theta}$

where $I = \frac{1}{2}Mr^2 \Rightarrow T.r - k.r^2\theta = \frac{1}{2}M.r^2\ddot{\theta}$

From equation (1) $T = -m\ddot{x}$

$\Rightarrow -m\ddot{x}r - kr^2\theta = \frac{1}{2}mr^2\ddot{\theta}$



But $x = r\theta$

$\Rightarrow \frac{1}{2}Mr^2\ddot{\theta} + mr^2\ddot{\theta} + kr^2\theta = 0$

$\Rightarrow (\frac{1}{2}M + m)\ddot{\theta} + k\theta = 0 \longrightarrow \therefore \omega_n = \left(\frac{k}{(\frac{1}{2}M + m)}\right)^{1/2}$

* By Energy Method.

$T =$ K.E. of the mass (linear) + K.E. of the pulley (Angular)

$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2$

$U =$ P.E. of the spring $= \frac{1}{2}kx^2 = \frac{1}{2}kr^2\theta^2$

$\frac{d}{dt}(T + U) = 0$

$\Rightarrow \frac{1}{2}mr^2.2\dot{\theta}\ddot{\theta} + \frac{1}{2}I.2\dot{\theta}\ddot{\theta} + \frac{1}{2}kr^2.2\theta\dot{\theta} = 0$

$mr^2\ddot{\theta} + \frac{1}{2}Mr^2\ddot{\theta} + kr^2\theta = 0$

$(m + \frac{1}{2}M)r^2\ddot{\theta} + kr^2\theta = 0 \quad] \div r^2$

$(m + \frac{1}{2}M)\ddot{\theta} + k\theta = 0$

$\Rightarrow \ddot{\theta} + \left(\frac{k}{m + \frac{1}{2}M}\right)\theta = 0$

$\therefore \omega_n = \left(\frac{k}{m + \frac{1}{2}M}\right)^{1/2} ; \quad f_n = \frac{1}{2\pi}\left(\frac{k}{m + \frac{1}{2}M}\right)^{1/2}$.

* Effective Mass: الكتلة المؤثرة

In general: the kinetic energy of any system is divided into:

1. Linear $\Rightarrow T = \frac{1}{2} m \dot{x}^2$
2. Rotational $\Rightarrow T = \frac{1}{2} I \dot{\theta}^2$

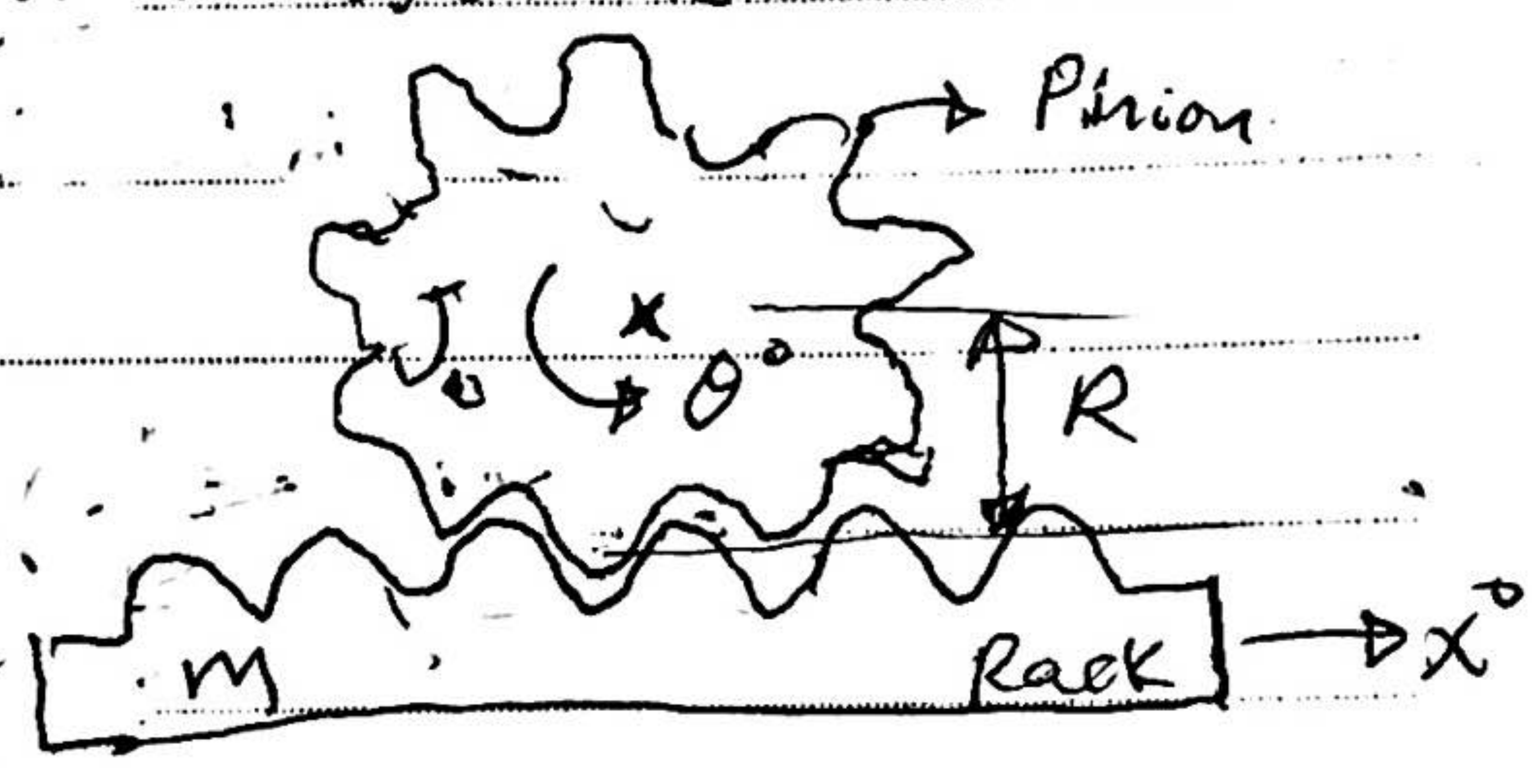
and can be written as

1. $T = \frac{1}{2} m_{eff} \dot{x}^2$
2. $T = \frac{1}{2} I_{eff} \dot{\theta}^2$

$\}$ for complex system

Example: Let a mass (m) having a translational velocity $\dot{x}$ be coupled to another mass (of mass moment of inertia J_0) having a rotational speed $\dot{\theta}$ as shown in figure. Determine the effective mass for the systems.

Sol:



$$T_{Rack} = \frac{1}{2} m \dot{x}^2$$

$$T_p = \frac{1}{2} J_0 \dot{\theta}^2$$

$$\frac{1}{2} m_{eff} \dot{x}^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

But $\dot{x} = \dot{\theta} * R \Rightarrow \dot{\theta} = \frac{\dot{x}}{R}$ sub in equation

$$\cancel{\frac{1}{2}} m_{eff} \cancel{\dot{x}^2} = \cancel{\frac{1}{2}} m \cancel{\dot{x}^2} + \cancel{\frac{1}{2}} J_0 \left(\frac{\cancel{\dot{x}^2}}{R^2}\right)$$

$$m_{eff} = \left(m + \frac{J_0}{R^2}\right) \quad \text{* ANS}$$

* Effective Stiffness.

The potential energy in any system can be divided into;

ط مرنة 1. potential energy in the spring $U = \frac{1}{2} k x^2$

ط وضعية 2. potential energy from the falling of the mass

$$U = W \cdot h = mgh$$

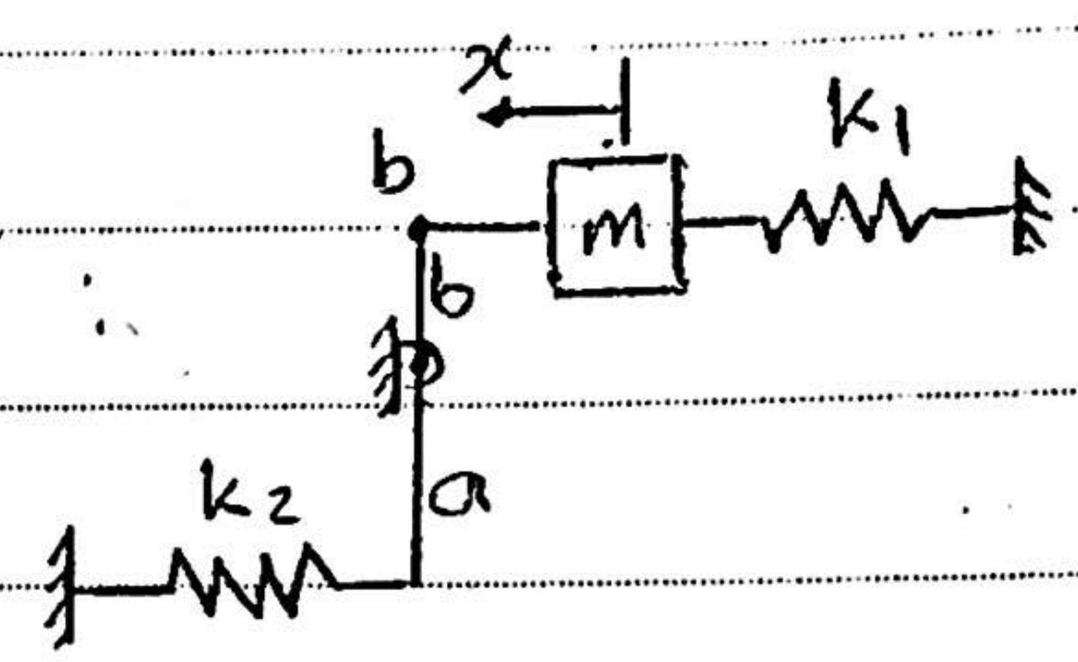
Where h: the hight.

An Complex system, the total potential energy;

$$U = \frac{1}{2} K_{eff} x^2$$

Example: Determine the effective stiffness of the system?

Solutions:



The potential energy of the system

$$U = \frac{1}{2} K_{eff} x^2$$

But the total potential energy is;

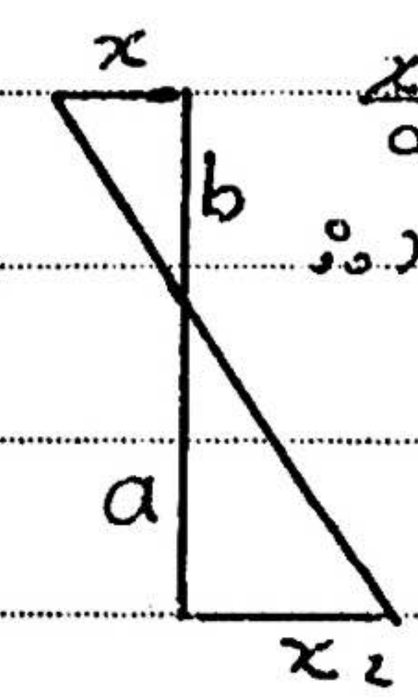
$$U_{act} = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 x_2^2$$

∴ the potential energy must be a function of (x), than

$$U_{act} = \frac{1}{2} K_1 x^2 + \frac{1}{2} K_2 \left(\frac{a}{b} x\right)^2$$

$$= \frac{1}{2} \left(K_1 + \frac{a^2}{b^2} K_2\right) x^2$$

$$= \frac{1}{2} K_{eff} x^2$$



$$\frac{x_2}{a} = \frac{x}{b}$$

$$\therefore x_2 = \frac{a}{b} x$$

$$\therefore \quad K_{eff} = K_1 + \frac{a^2}{b^2} K_2$$

→ Continue أقلب الصفحة

Stiffness and Flexibility.

The stiffness can be defind as the force required to achive a unity displacement in a certain direction.

$\therefore F = K \cdot x \Rightarrow K = F/x$

There is a Combination, in some systems, at several linear springs. These springs can be combined into an equivalent single spring as explained below;

1. Spring in parallel.

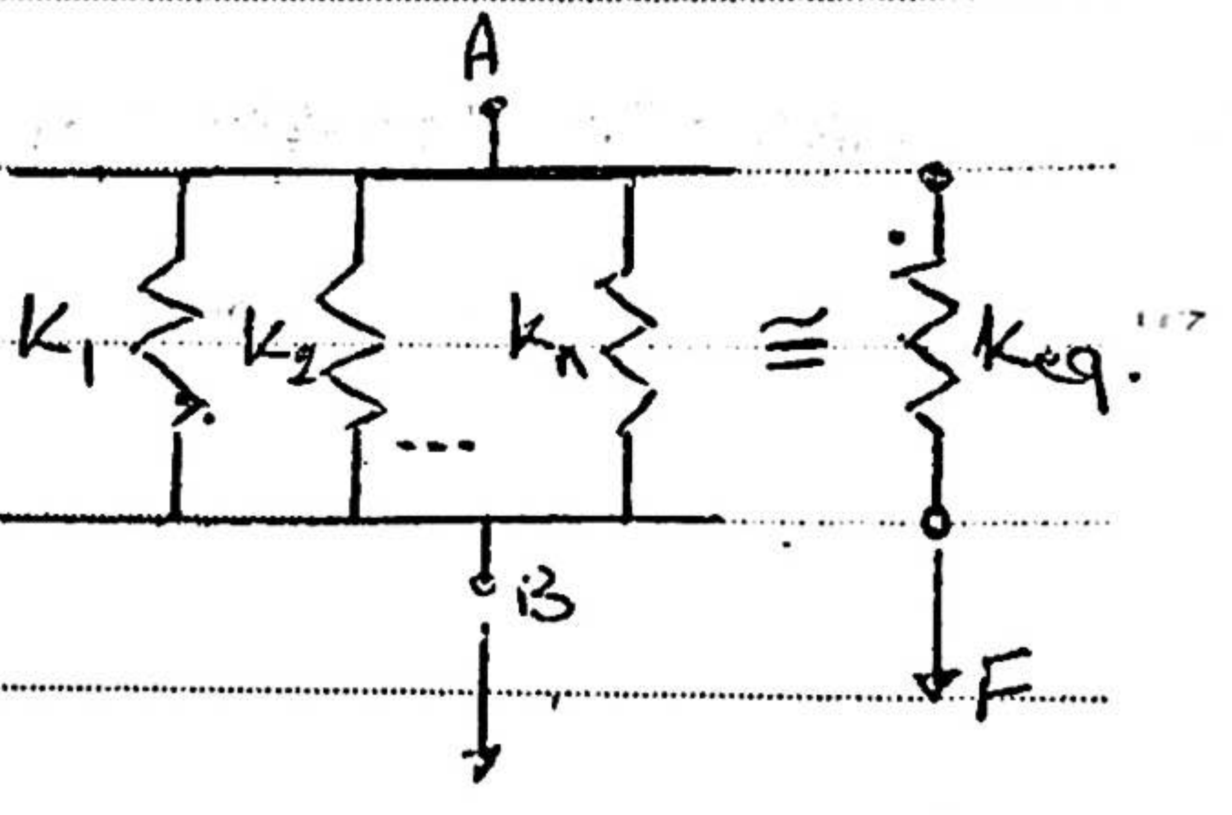
The spring constant of the equivalent spring is the sum of the spring constants of the springs (added forces with equal displacements).

$\therefore F = F_1 + F_2 + \dots + F_n$

$\Rightarrow F = k_1 x + k_2 x + \dots + k_n x$

$\therefore \frac{F}{x} = k_1 + k_2 + \dots + k_n = k_{eq}$

$\| k_{eq} = k_1 + k_2 + \dots + k_n \|$



2. Spring in series.

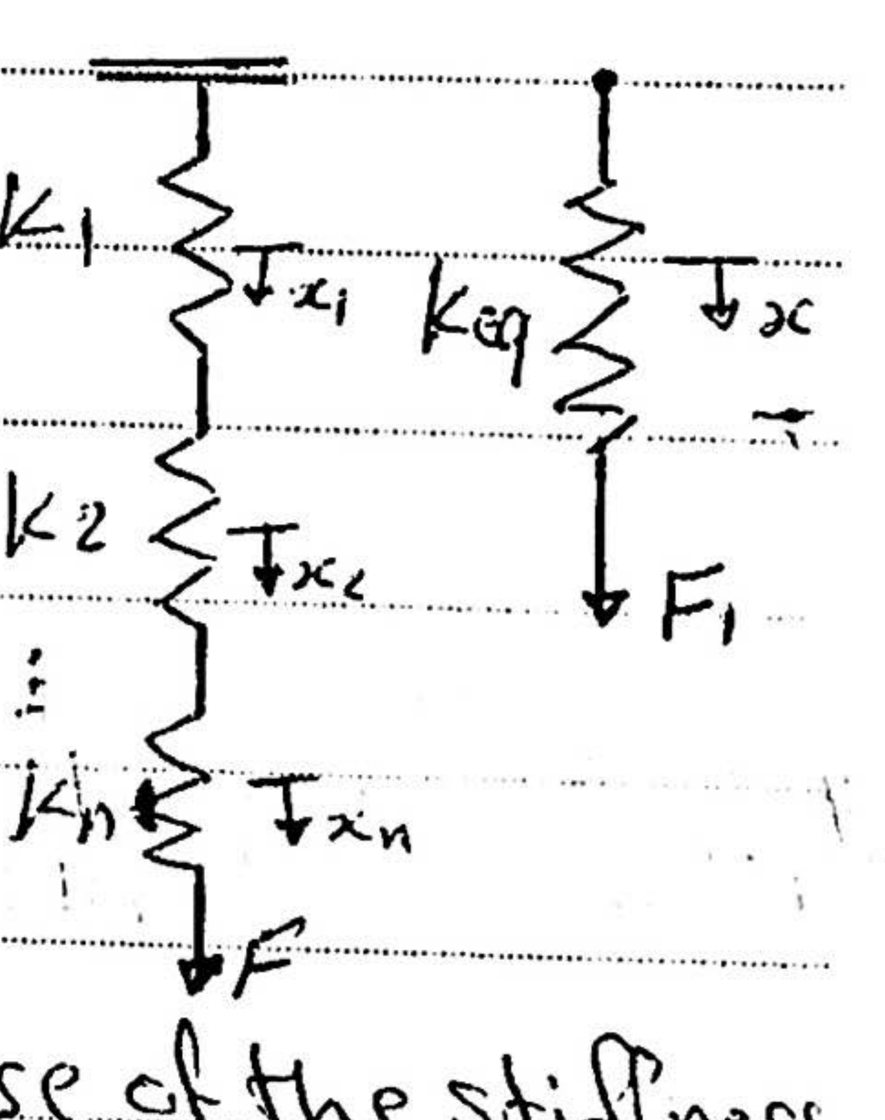
The spring constant of the equivalent spring is (added displacements with equal forces).

$\therefore x = x_1 + x_2 + \dots + x_n$

$= (F/k_1) + (F/k_2) + \dots + (F/k_n)$

$\Rightarrow \frac{x}{F} = \frac{1}{k_1} + \frac{1}{k_2} + \dots + \frac{1}{k_n} = \frac{1}{k_{eq}}$

$\left\| \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \dots + \frac{1}{k_n} \right\|$



* The flexibility is defined as the inverse of the stiffness.

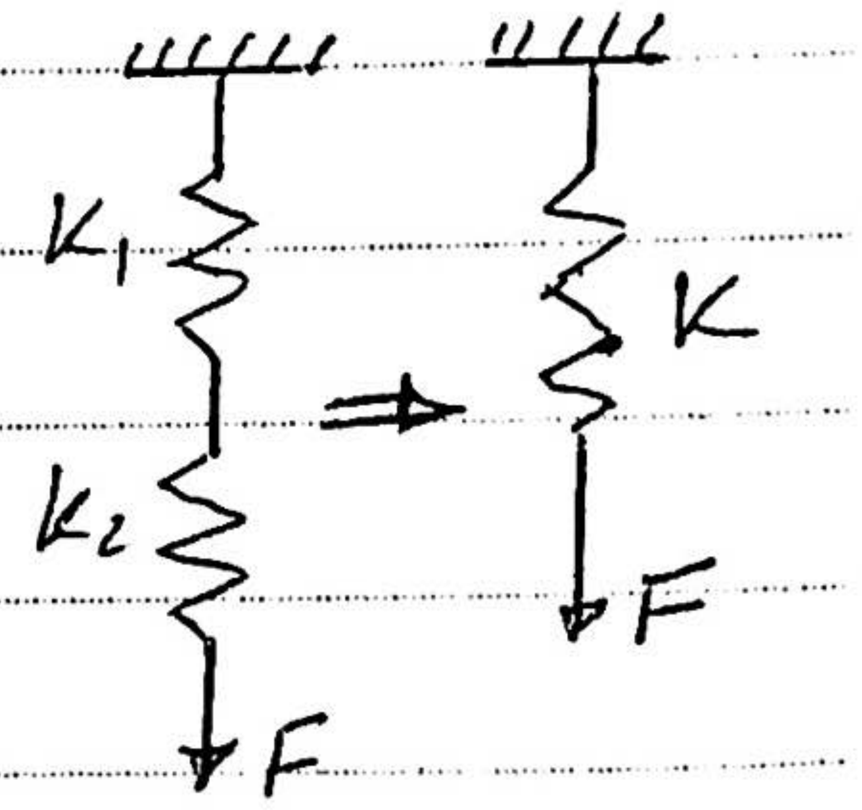
$\| a = \frac{1}{k} = \frac{x}{F} \|$

Example: Determine the spring stiffness for the system of springs.

(1) $F = F_1 = F_2$

$x = x_1 + x_2$

But $k = \frac{F}{x} \Rightarrow x = \frac{F}{k}$



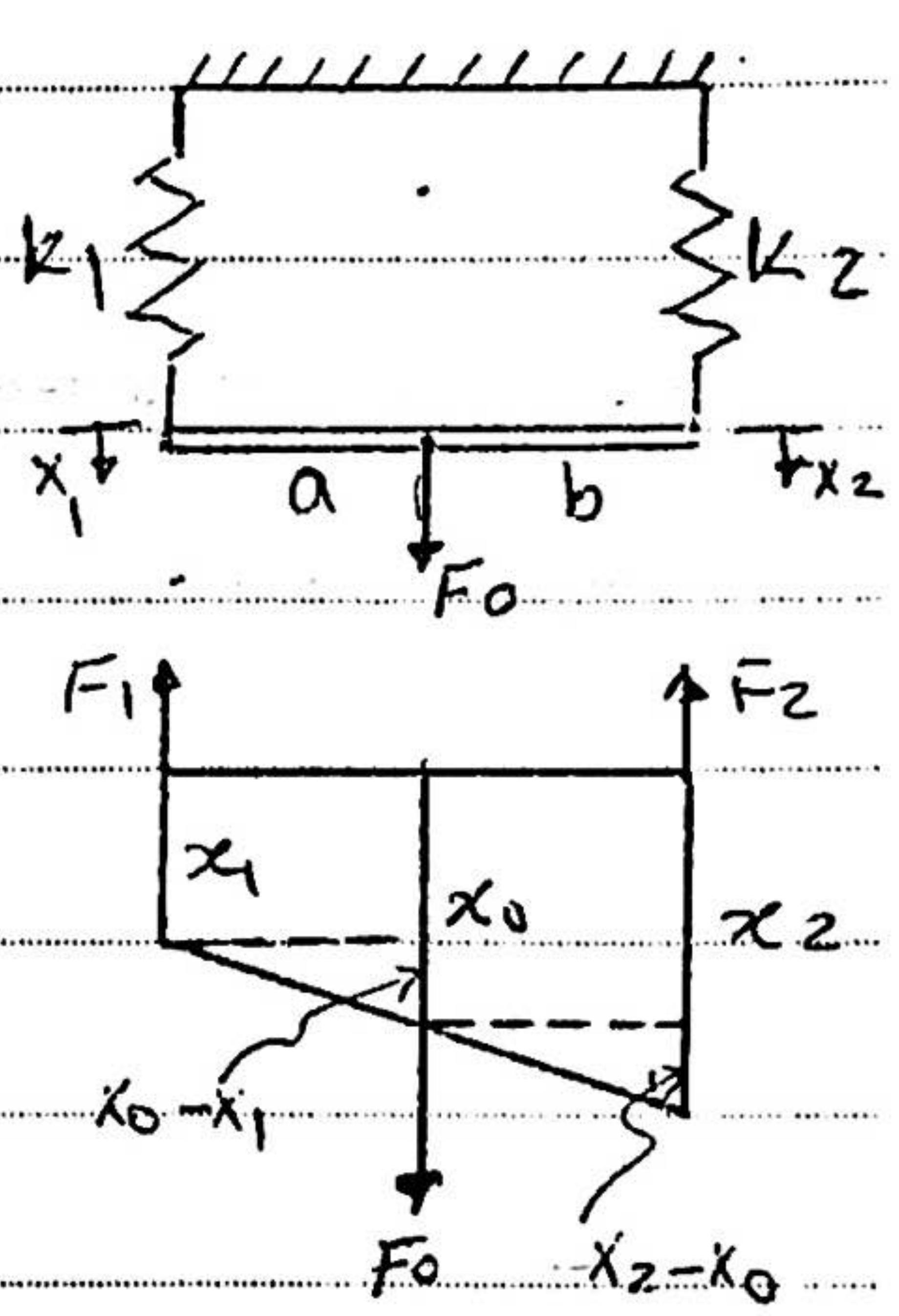
$\therefore \frac{F}{k} = \frac{F_1}{k_1} + \frac{F_2}{k_2}$

$\Rightarrow \frac{F}{k} = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$

$\Rightarrow k = (k_1 * k_2)/(k_1 + k_2)$

(2) $F_1 = k_1 x_1$, $F_2 = k_2 x_2$

$F_o = k_o \cdot x_o$, $k_o = ?$



$\sum M_2 = 0$

$F_1 \cdot (a+b) = F_o \cdot b \Rightarrow F_1 = \frac{b}{a+b} F_o$

$\sum M_1 = 0$

$F_2 (a+b) = F_o \cdot a \Rightarrow F_2 = \frac{a}{a+b} F_o$

$x_1 = \frac{F_1}{k_1} = \frac{b F_o}{(a+b) k_1} = \frac{b}{a+b} \cdot \frac{F_o}{k_1}$

$x_2 = \frac{F_2}{k_2} = \frac{a F_o}{(a+b) k_2} = \frac{a}{a+b} \cdot \frac{F_o}{k_2}$

$\frac{x_o - x_1}{a} = \frac{x_2 - x_o}{b}$

$\frac{1}{a}\left[\frac{F_o}{k_o} - \frac{b}{a+b} \cdot \frac{F_o}{k_1}\right] = \frac{1}{b}\left[\frac{a}{a+b} \cdot \frac{F_o}{k_2} - \frac{F_o}{k_o}\right]$

$k_o = \frac{(a+b)^2}{\frac{a^2}{k_2} + \frac{b^2}{k_1}}$

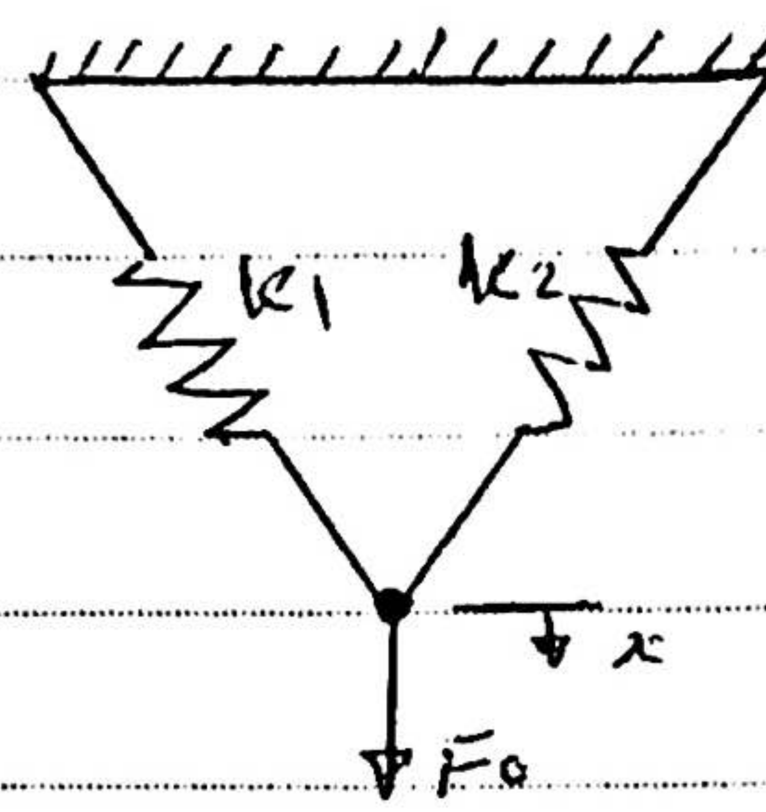
* spesial Case , $a = b \Rightarrow k_o = \frac{4 k_1 k_2}{k_1 + k_2}$

Note / Try to solve it by energy method

(3) $F_1 = k_1 x$, $F_2 = k_2 x$, $F_o = k_{eq}\, x$

$\because F_o = F_1 + F_2 \Rightarrow k_{eq} \cdot x = k_1 x + k_2 x$

$\Rightarrow k_{eq} = k_1 + k_2$



OR.

$\because U = \frac{1}{2} k x$

$\therefore U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$

But $x = x_1 = x_2$

$\therefore U = \frac{1}{2} (k_1 + k_2) x^2$

$\therefore k_{eq} = k_1 + k_2$

Viscously Damped Free Vibration

Viscous damping force is expressed by the equation

$$F_d = c\dot{x} \quad ......(1)$$

where c is a constant of proportionality.

Symbolically. it is designated by a dashpot, as shown in Figure 1. From the free body diagram, the equation of motion is. seen to be

$$m\ddot{x} + c\dot{x} + kx = F(t) \quad ......(2)$$

The solution of this equation has two parts. If F(t) = 0, we have the homogeneous differential equation whose solution corresponds physically to that of free-damped vibration. With F(t) = 0, we obtain the particular solution that is due to the excitation irrespective of the homogeneous solution. We will first examine the homogeneous equation that will give us some understanding of the role of damping.

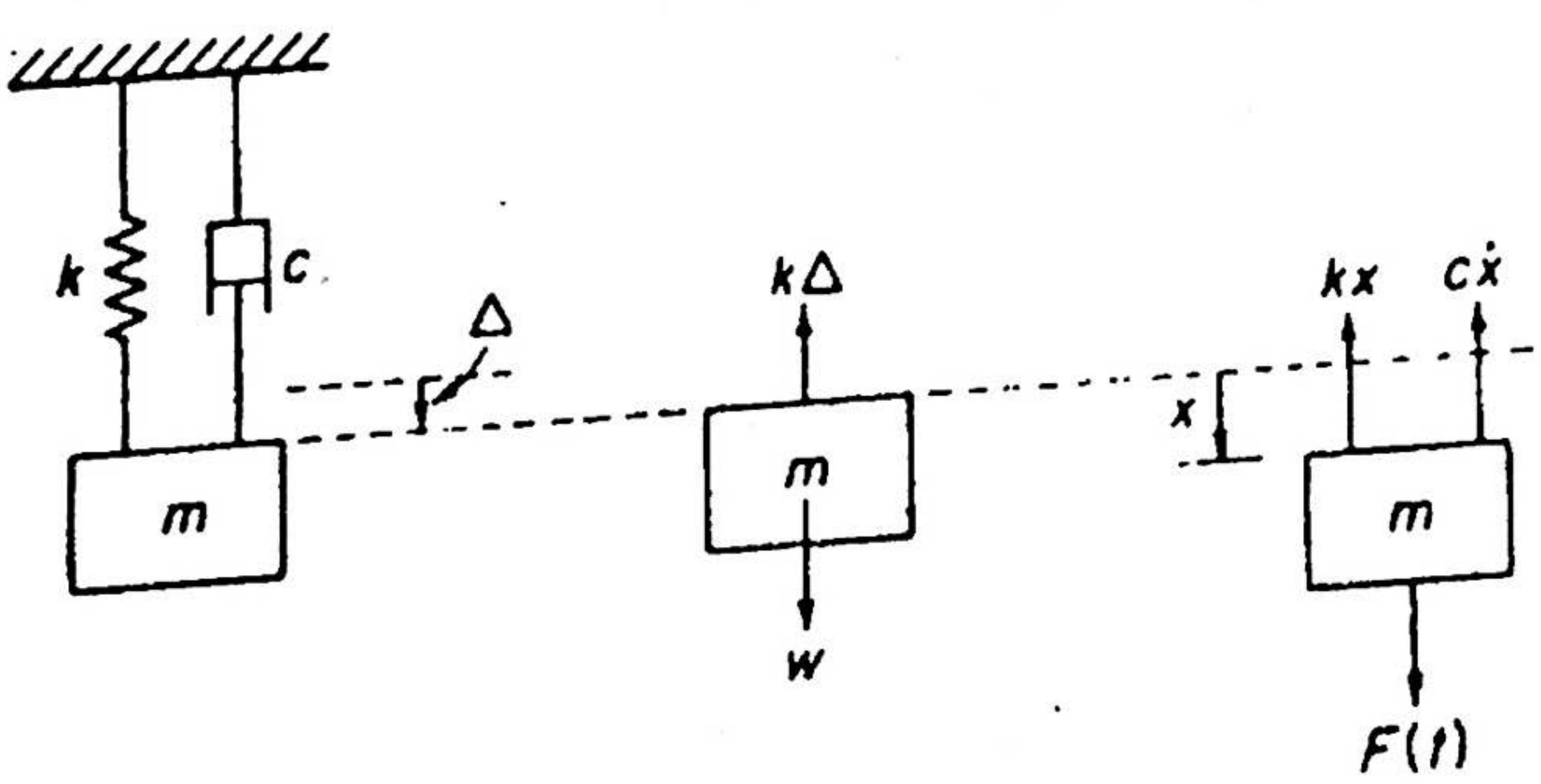


Figure .1 Viscously Damped Free Vibration

With the homogeneous equation :

$$m\ddot{x} + c\dot{x} + kx = 0 \quad ......(3)$$

the traditional approach is to assume a solution of the form :

$$x = e^{st} \quad ......(4)$$

Where s is a constant. Upon substitution into the differential equation, we obtain :

$$\left(ms^2+cs+k\right)e^{st}=0$$

Which is satisfied for all values of t when

$$s^2+\frac{c}{m}s+\frac{k}{m}=0 \qquad ......(5)$$

Equation (5), which is known as the characteristic equation, has two roots :

$$s_{1,2}=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}} \qquad ......(6)$$

Hence, the general solution is given by the equation:

$$\boxed{x(t)=Ae^{s_1t}+Be^{s_2t} \qquad ......(7)}$$

$\hookrightarrow x(t)$

Where A and B are constants to be evaluated from the initial conditions $x(0)$ and $\dot{x}(0)$. Equation (6) substituted into (7) gives :

$$\boxed{x=e^{-(c/2m)t}\left(Ae^{\left(\sqrt{(c/2m)^2-k/m}\right)t}+Be^{-\left(\sqrt{(c/2m)^2-k/m}\right)t}\right) \qquad ......(8)}$$

The first term, $e^{-(c/2m)t}$, is simply an exponentially decaying function of time. The behavior of the terms in the parentheses, however, depends on whether the numerical value within the radical is positive, zero, or negative.

When the damping term (c/2m)2 is larger than k/m, the exponents in the previous equation are real numbers and no oscillations are possible. We refer to this case as over damped.

$\hookrightarrow$ because $\omega_n=\sqrt{\frac{K}{m}}$

When the damping term $(c/2m)^2$ is less than k/m, the exponent becomes an imaginary number, $\pm i\sqrt{k/m-(c/2m)^2}\,t$. Because

$$e^{\pm i\left(\sqrt{k/m-(c/2m)^2}\right)t}=\cos\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\,t\pm i\sin\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}\,t$$

The terms of Equation (8) within the parentheses are oscillatory. We refer to this case as under damped.

In the limiting case between the oscillatory and non oscillatory motion $(c/2m)^2=k/m$, and the radical is zero. The damping corresponding to this case is called critical damping, cc.

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\,\omega_n = 2\sqrt{km} \qquad \ldots\ldots(9)$$

مهم

Any damping can then be expressed in terms of the critical damping by a non dimensional number ζ, called the damping ratio:

$$\zeta=\frac{c}{c_c} \qquad \ldots\ldots(10)$$

مهم

and

$$\frac{c}{2m}=\zeta\left(\frac{c_c}{2m}\right)=\zeta\omega_n \qquad \ldots\ldots(11)$$

مهم

⇒ ملاحظة مهمة - للمقارنة

التردد الاهتزازي (ω_d) للاخماد

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$

ثابت

معامل الاخماد

أقل

<u>(i) Oscillatory Motion ($\zeta < 1.0$) Under damped Case :</u>

$$x=e^{-\zeta\omega_n t}\left(Ae^{i\left(\sqrt{1-\zeta^2}\right)\omega_n t}+Be^{-i\left(\sqrt{1-\zeta^2}\right)\omega_n t}\right) \qquad \ldots\ldots(12)$$

ω_d

The frequency of damped oscillation (damped natural frequency) is equal to :

$$\omega_d = \frac{2\pi}{\tau_d} = \omega_n\sqrt{1-\zeta^2} \qquad \ldots\ldots(13)$$

In this case the square root of (ζ^2-1) is a imaginary number and can be written as $(\pm\zeta^2-1)$. Such a system is called an under damping system and will vibrate when released.

Figure 2. shows the general nature of the oscillatory motion.

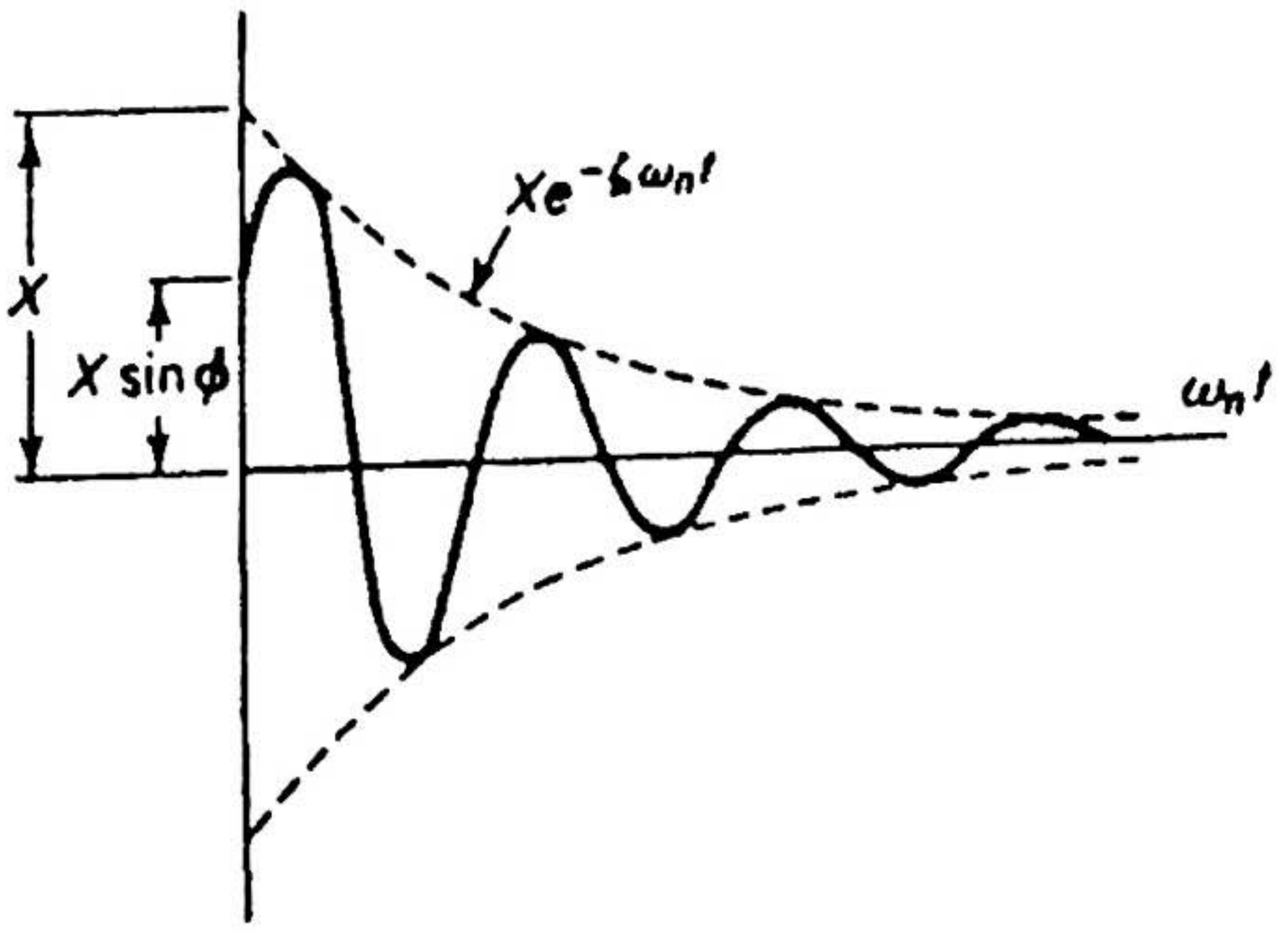


Figure 2. Damped Oscillation $\zeta < 1$

(ii) Non oscillatory Motion ($\zeta > 1.0$) Overdamped Case :

$$x = Ae^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + Be^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \quad \ldots\ldots(14)$$

The motion is an exponentially decreasing function of time as shown in Figure 3.

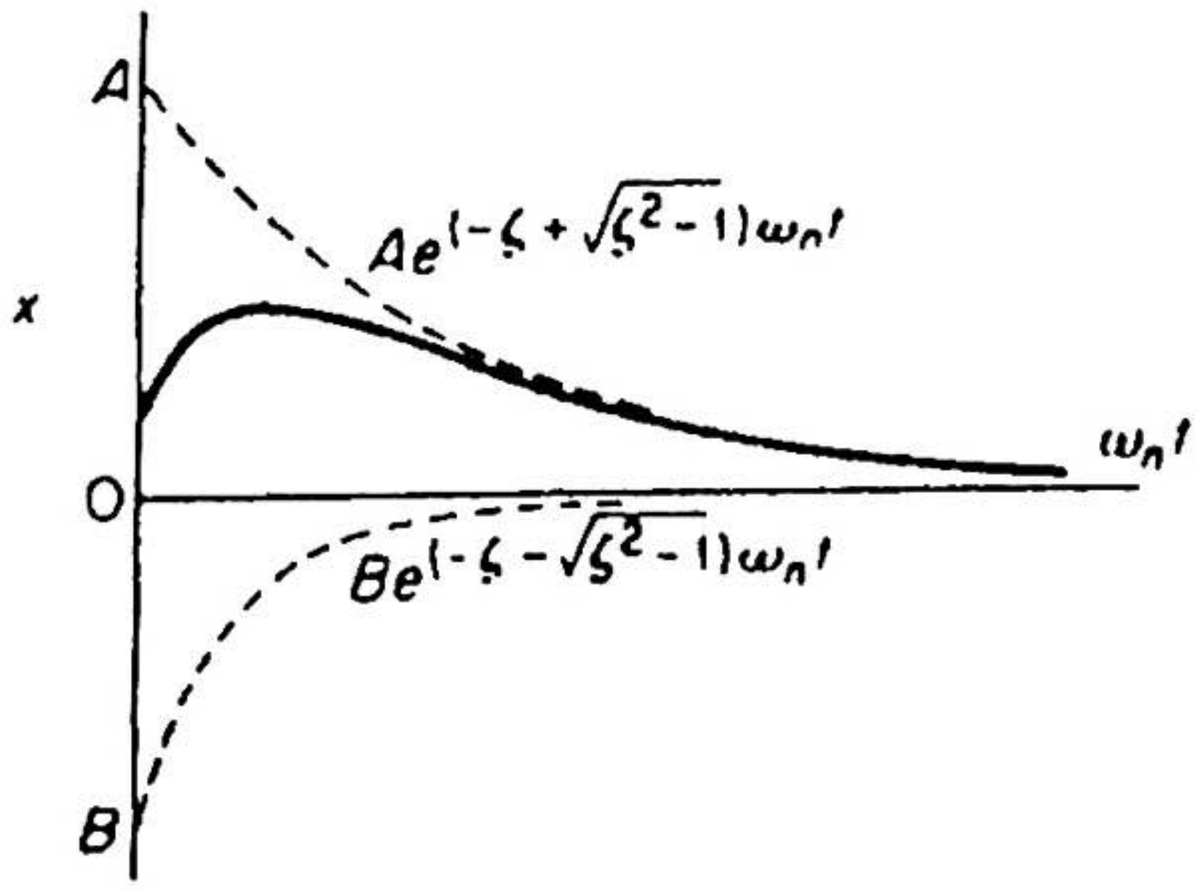


Figure 3. A periodic Motion $\zeta > 1$

In this case the square root of $(\zeta^2 - 1)$ is a real number and equation (12) indicates that the displacement will vary exponentially with time. There will be no oscillatory motion – no vibration. This is known as an overdamping system.

(iii) Critically Damped Motion ($\zeta = 1.0$) :

This is known as a critically damped system, and represents the case where the system is just non-oscillatory. The displacement gradually returns to its initial value, with no vibration. It can be shown that the solution in this case is:

$$x = (A + Bt)e^{-\omega_n t} \qquad ......(15)$$

Figure 4. shows three types of response with initial displacement x(0).

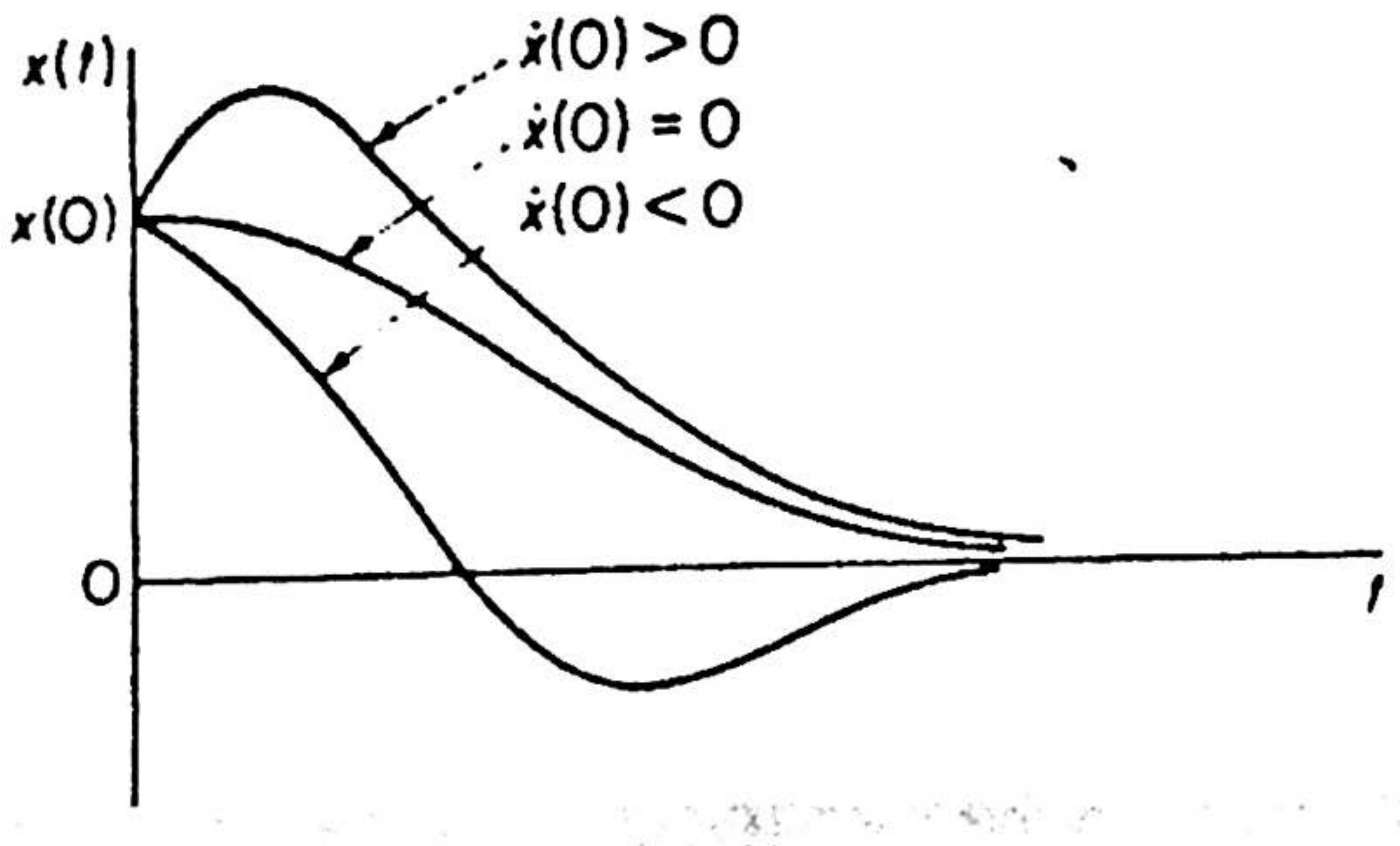


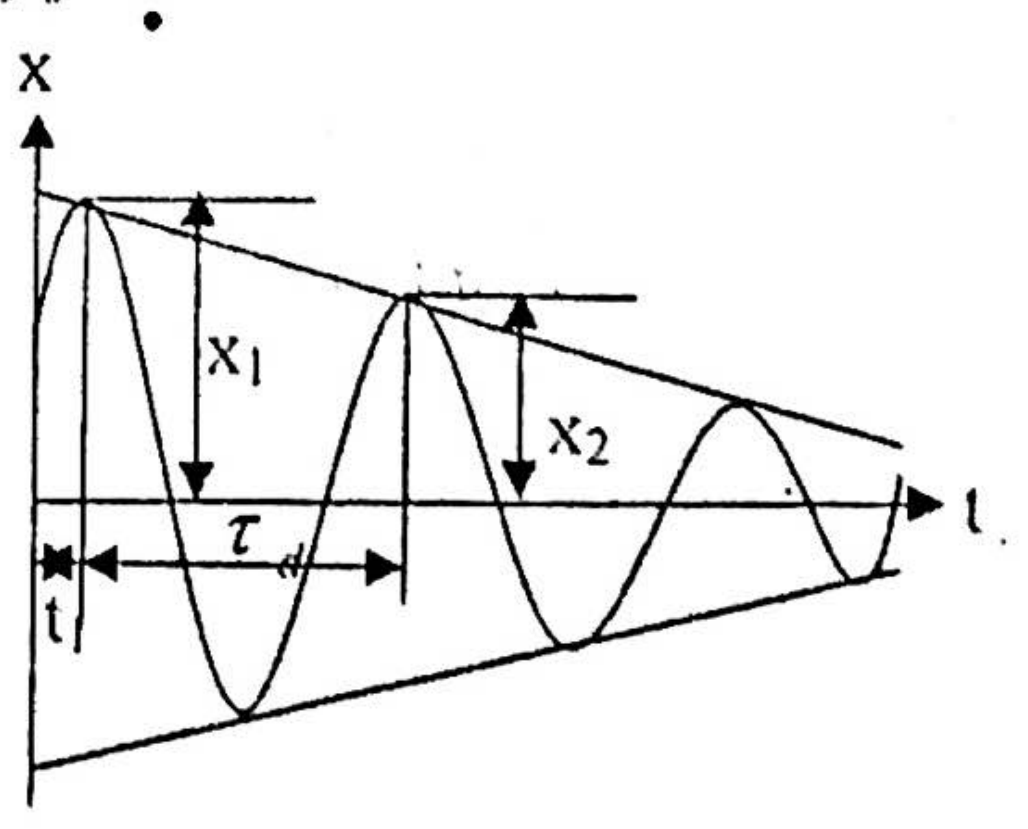
Figure 4. Critically Damped Motion $\zeta = 1$

أكبر

* To summaries, if $\zeta > 1$, the system is heavily damped, and no vibration occurs. If $\zeta = 1$, the system is critically damped, and the damping is only just sufficient to prevent vibration. If $\zeta < 1$, as it is in the majority of cases, there is in sufficient damping in the system to prevent vibration and the motion is oscillatory.

Logarithmic Decrement

The rate at which the amplitude decays gives us another measurement of the damping in a system, known as the *Logarithmic Decrement*, δ. This is defined as the natural logarithm of the ratio of any two successive amplitudes. In general, we have vibration at a frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$, so the time for one cycle is $T = 2\pi/\omega_d = 2\pi/\omega_n\sqrt{1-\zeta^2}$. The amplitude of the vibration is $xe^{-\zeta\omega_n t}$.



$$\delta = \ln x_1/x_2$$

Let δ = Logarithmic Decrement متتاليه
Where : x_1 and x_2 are two successive amplitudes.

$$\delta = \zeta\omega_n\tau_d \quad , \quad \tau_d = \frac{2\pi}{\omega_n}\sqrt{1-\zeta^2} \quad , \quad \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} .$$

$\Rightarrow 0 = X(3.6\cos\phi - (0.8\times 6)\sin\phi)$ --(2)

From (2), $3.6\cos\phi = 4.8\sin\phi$

because X does not equal 0.

$\therefore \tan\phi = \frac{3.6}{4.8} = 0.75 \Rightarrow \phi = 36.9^\circ = 0.644$ rad.

putting this value for ϕ in (1) $\Rightarrow X = \frac{0.01}{\sin 36.9^\circ} = 0.017$ m.

The displacement is, therefore.

$x = X e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$

$= 0.017\, e^{-4.8t}\sin(3.6t + 0.644)$ معادلة الحركة النهائية

The amplitude of the response is given by the exponential term

$$X e^{-\zeta\omega_n t} = (0.017 e^{-4.8t})$$

So, if the amplitude is 1mm = 0.001 m, then;

$0.001 = 0.017 e^{-4.8t}$

$\therefore e^{-4.8t} = \frac{0.001}{0.017} = 0.0588$

$\delta = \ln\frac{x_1}{x_2} = \ln\frac{0.017}{0.001}$

$\delta = 2.8333$

$\delta = \zeta\omega_n\tau_d \Rightarrow \tau_d = \frac{2.8333}{4.8}$

Taking the natural log of both sides of this equation,

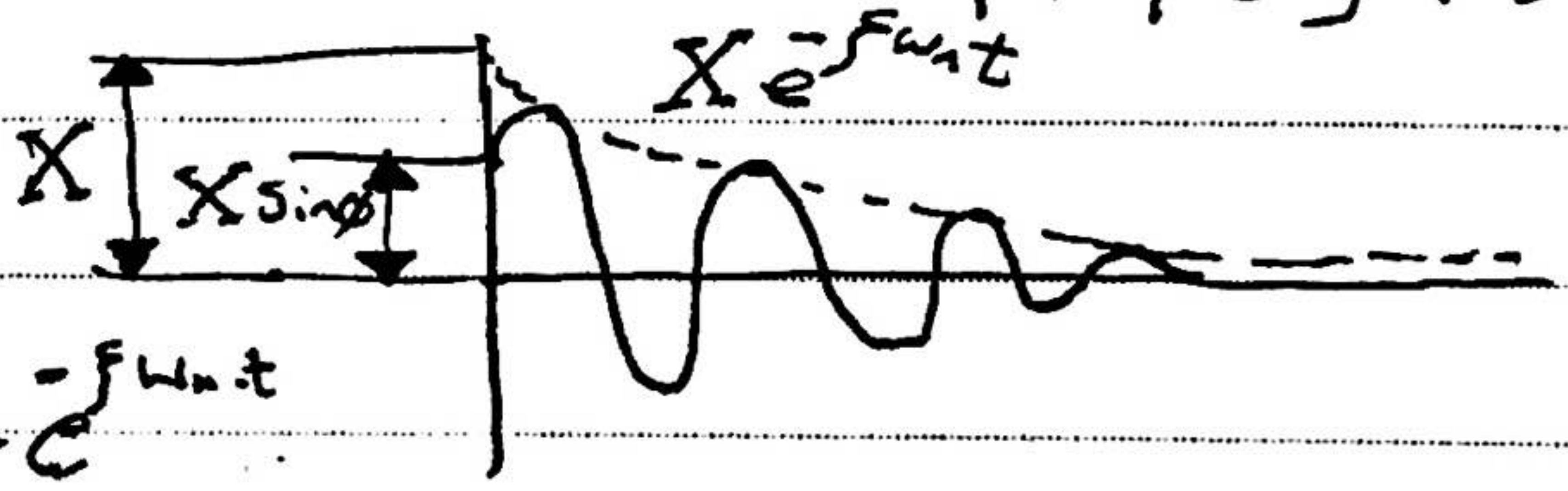
$-4.8t = \ln(0.0588) = -2.836$

$\Rightarrow t = 0.59$ sec

$\therefore \tau_d = 0.59$ sec

The amplitude will become less than 1mm, therefore, after 0.59 seconds only.

ملاحظة / لحساب قيمة الزمن اللازم لتناقص السعة نستخدم معادلة التناقص اللزجي أي:-



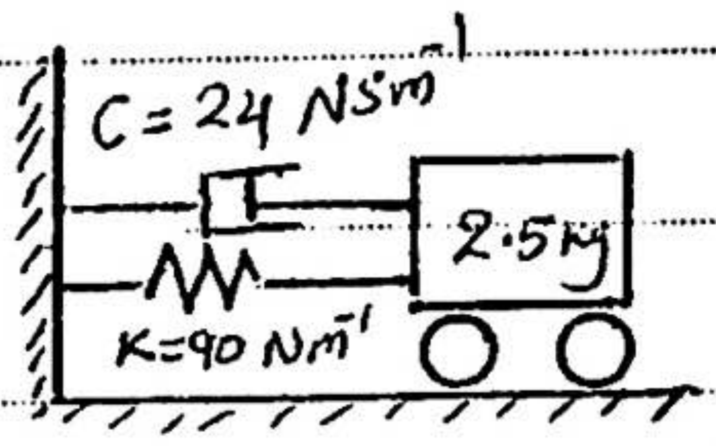
$X = X\cdot e^{-\zeta\omega_n t}$

يراجع

Examples (1) Determine the value of the natural frequency and the damping ratio for the simple spring - mass - dashpot system shown. (2) If the mass is released from rest at a distance (10 mm) to the right of the equilibrium position. Determine the displacement as a function of time, and estimate how long it will take for the amplitude of the motion to be reduced to less than (1 mm).

تقل

Solution:



(1) $\omega_n = (K/m)^{1/2} = (90/2.5) = 6\ rad\ sec^{-1}$

$\xi = (c/(2\sqrt{km})) = 24/(2*\sqrt{90*2.5})$

$= 0.8$

So the mass will oscillate if given an initial displacement and released, but the motion will be damped out quite quickly, since ξ is not much less than (1).

(2) Since $\xi < 1$, the system is underdamped. The solution to the equation of motion is therefor given by equation

$x = X e^{-\xi\omega_n t} \sin(\omega_d t + \phi)$

Where the damped frequency $\omega_d = \omega_n\sqrt{1-\xi^2}$

The initial conditions are that at $t = 0$, $x = 10\ mm = 0.01\ m$ and $\dot{x} = 0$,

$\therefore \omega_n = 6\ rad\ sec^{-1} \Rightarrow \omega_d = 6 * \sqrt{1 - 0.8^2} = 3.6\ rad\ sec^{-1}$

Differentiating with respect to time, we find that: معادلة رقم اثنين

$\dot{x} = X[\omega_d e^{-\xi\omega_n t}\cos(\omega_d t + \phi) - \xi\omega_n e^{-\xi\omega_n t}\sin(\omega_d t + \phi)]$

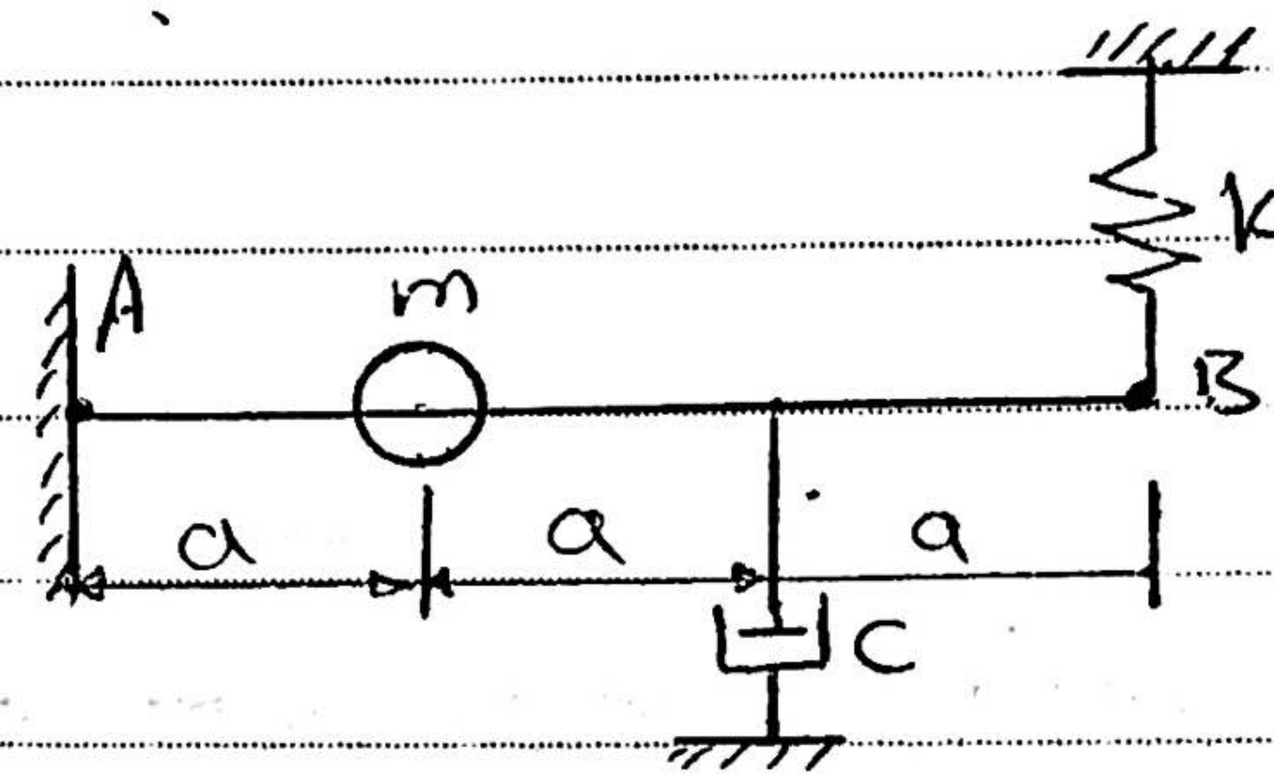
So, putting in the initial conditions:

For x: $0.01 = X \sin\phi$ ---(1) معادلة رقم واحد

For $\dot{x}$: $0 = X[\omega_d \cos\phi - \xi\omega_n \sin\phi]$

Example: A rod is hinged at one end and supported by a spring of stiffness "k" at the other end. A mass of "m" is attached at "l/3" length from the hinge and a dashpot having a damping coefficient "c" is attached "2l/3" of length from the hinge. Find the equivalent mass at B and damping coefficient at the spring and derive an expression for the frequency of the damped free vibrations of the system. as shown in fig

solution:



Neglecting the mass of the rod, spring and dashpot,

$I_A = m a^2$

let x be the instant-teneous displacement of end B

* Note / $I_A = m.a^2$

Then restoring force exerted by

$= kx$

Restoring moment about A

$= kx * 3a$

Velocity of dashpot $\frac{2}{3}\frac{dx}{dt}$

∴ damping force $= c.\frac{2}{3}\frac{dx}{dt}$

damping moment about A $= c * \frac{2}{3}\frac{dx}{dt} * 2a = \frac{4}{3}ac\frac{dx}{dt}$

Angular acceleration of rod about A $= \left(\frac{d^2x}{dt^2}\right)/3a$

The equation of motion of the rod:

$$\frac{ma^2}{3a}\left(\frac{d^2x}{dt^2}\right) + 4ac\frac{dx}{dt} + 3akx = 0$$

dividing through by 3a gives:

$$\frac{m}{9}\left(\frac{d^2x}{dt^2}\right) + \frac{4C}{a}\frac{dx}{dt} + kx = 0$$

$\therefore$ equivalent mass at spring $= \frac{m}{9}$, and equivalent damping coefficient at spring $= \frac{4C}{a}$

$\therefore \omega_n^2 = \frac{k}{m}$ generally $\Rightarrow$ for this system $\omega_n^2 = \frac{9k}{m}$

and generally $\xi = \frac{c}{2\sqrt{km}}$;

for this system $\Rightarrow \xi = \frac{4C}{9 * 2\sqrt{km/9}} = \frac{2C}{3\sqrt{km}}$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$= 3\sqrt{\frac{k}{m}}\sqrt{1 - \frac{4C^2}{9km}}$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{9k}{m} - \frac{9 * 4C^2 k}{9km^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{9k}{m} - \frac{4C^2}{m^2}}$$

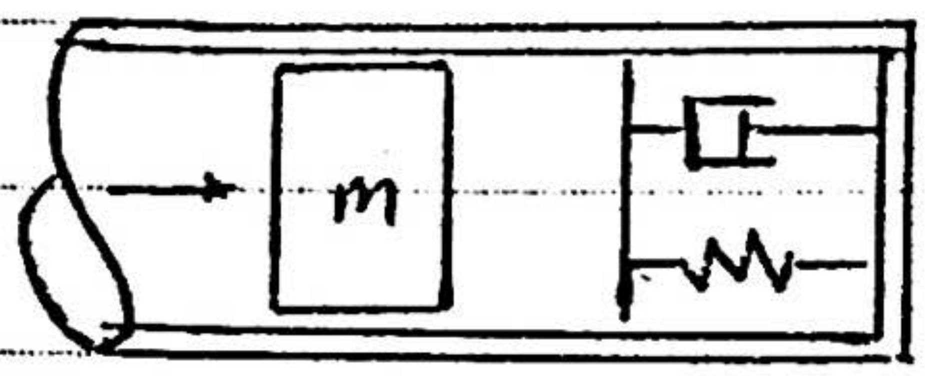
$$= \frac{1}{2\pi m}\sqrt{9km - 4C^2} \quad HZ$$

on student

Example: A piston of mass (4.53 kg) is traveling in a tube with a velocity of (15.24 m/s) and engages a spring and damper as shown. Determine the maximum displacement or ampilitude of the piston after engaging the spring-damper.

Solution:

$m\ddot{x} + c\dot{x} + kx = 0$



$\therefore$ the solution of this equation of motion is,

$x(t) = X \cdot e^{-\xi w_n t} \cdot \sin(\sqrt{1-\xi^2} \cdot w_n t + \phi)$

$\dot{x}(t) = X [e^{-\xi \cdot w_n t} \cdot \sqrt{1-\xi^2} \cdot w_n \cdot \cos(\sqrt{1-\xi^2} \cdot w_n t + \phi)$
$- \xi \cdot w_n \cdot e^{-\xi \cdot w_n \cdot t} \cdot \sin(\sqrt{1-\xi^2}\, w_n t + \phi)$

* at $t=0$, $x(0)=0 \Rightarrow 0 = X \cdot \sin\phi \Rightarrow \therefore \phi = 0$
* at $t=0$, $\dot{x}(0) = 15.24$ m/s $\Rightarrow 15.24 = X(\sqrt{1-\xi^2} \cdot w_n)$

But $w_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{35000}{4.53}} = 87.899$ rad/s

$\xi = \frac{c}{2 \cdot m \cdot w_n} = \frac{1.75 * 100}{2(4.53)(87.899)} = 0.2198$

$\Rightarrow 15.24 = X(\sqrt{1-0.2198^2})(87.899)$

$\therefore X = 0.1777$ m.

H.W

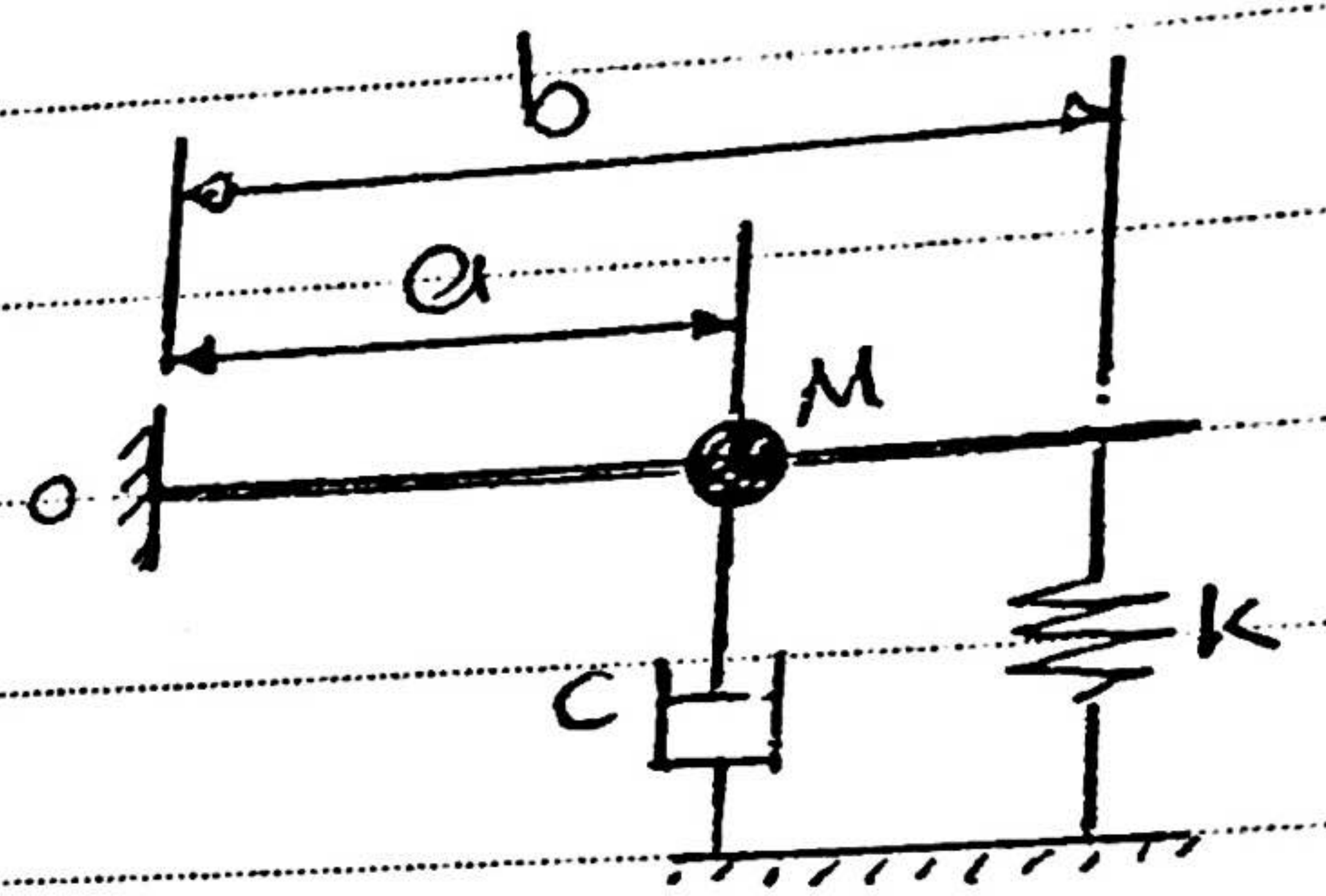
$\therefore X(t) = 0.1777 \cdot e^{-\xi w_n \cdot t} \cdot \sin(\sqrt{1-\xi^2}\, W_n \cdot t + \phi)$

Note $\sin\phi = 0 \Rightarrow \phi = 0$

$\therefore X(t) = 0.1777 \cdot e^{-\xi w_n \cdot t} \cdot \sin\sqrt{1-\xi^2}\, w_n \cdot t$ ⟶ *

ملاحظة / أقصد إذا لم يعطها الجسم سرعة تساوي صفراً

Example: write the differential equation of motion for the system shown, and determine the natural frequency of damped oscillation and the critical damping coefficient.



Solution:

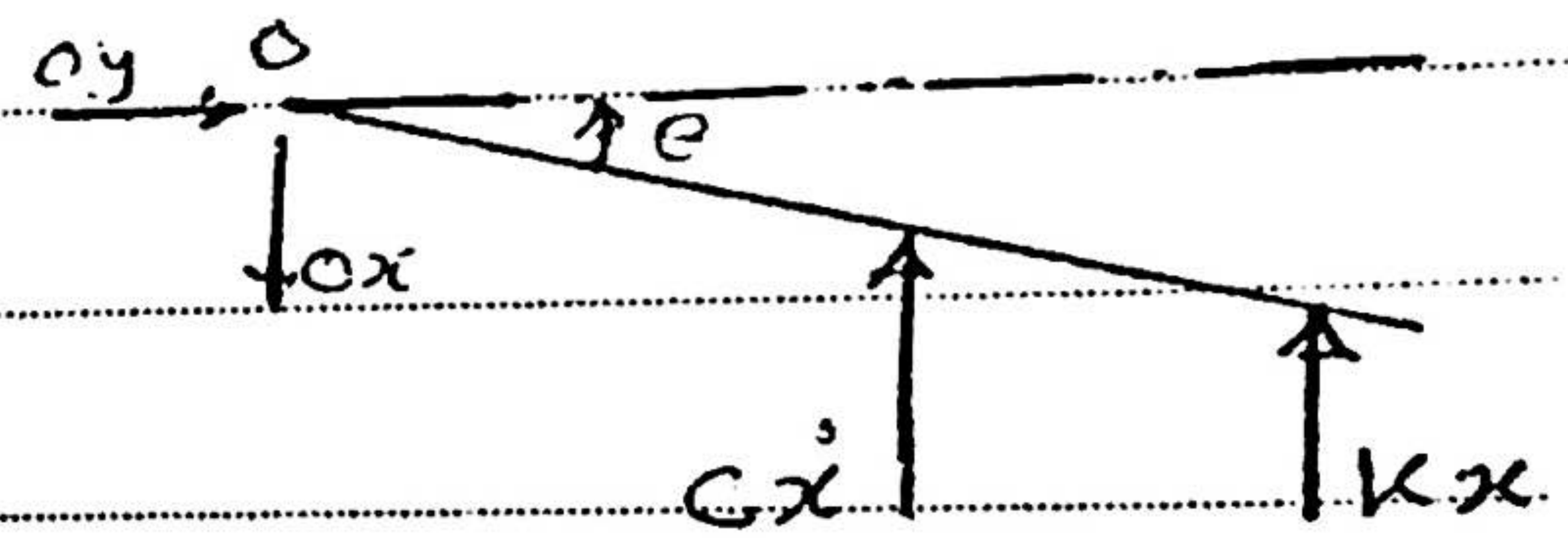
$\circlearrowleft + \; \sum M_o = I_o \cdot \ddot{\theta} \Rightarrow -c\dot{x}\cdot a - kx\cdot b = M a^2 \ddot{\theta}$

$\Rightarrow \quad M a^2 \ddot{\theta} + c\dot{x} a + k x b = 0$

$\Rightarrow \quad M a^2 \ddot{\theta} + C a^2 \dot{\theta} + k b^2 \theta = 0$

$\Rightarrow \quad M \ddot{\theta} + C \dot{\theta} + k \dfrac{b^2}{a^2} \theta = 0$

$\Rightarrow \quad \ddot{\theta} + \dfrac{C}{M} \dot{\theta} + \dfrac{k}{M} \cdot \dfrac{b^2}{a^2} \theta = 0$



OR $\quad \ddot{\theta} + 2\xi \omega_n \dot{\theta} + \omega_n^2 \theta = 0$

$\therefore \quad \omega_n = \left(\dfrac{k}{M} \cdot \dfrac{b^2}{a^2} \right)^{1/2}$

$2 \xi \omega_n = \dfrac{C}{M} \Rightarrow \xi = \dfrac{C}{2M\sqrt{\frac{k}{m} \cdot \frac{b^2}{a^2}}}$

But $\quad \omega_d = \sqrt{1-\xi^2}\, \omega_n \Rightarrow \therefore f_d = \omega_d / 2\pi = \dfrac{1}{2\pi} \sqrt{\left(\dfrac{b}{a}\right)^2 \cdot \dfrac{k}{M} - \left(\dfrac{C}{2M}\right)^2}$

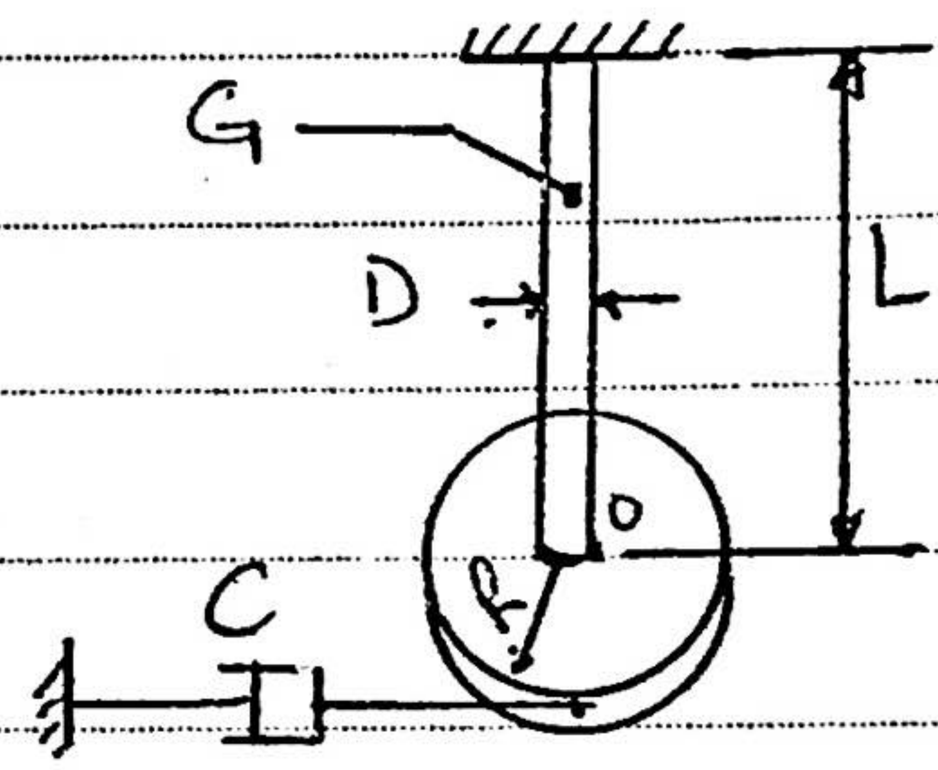
$\xi = \dfrac{C}{C_c} \Rightarrow$

$$C_c = \frac{C}{\xi} = 2M \left(\frac{b^2}{a^2} \cdot \frac{k}{M} \right)^{1/2} = \frac{2b}{a} (k \cdot M)^{1/2}$$

*

Example: The disc of mass (m) and (R) see in fig below is supported by an elastic shaft of diameter (D) and length (L). The elastic properties of the shaft are determined by the shear modulus (G). The disc can oscillate about the vertical axis and the damping is modelled by the linear damper of a coefficient (C). Produce equation of motion of the disc and determine the natural frequency.

solution:



$$\sum M_o = I\ddot{\theta}$$

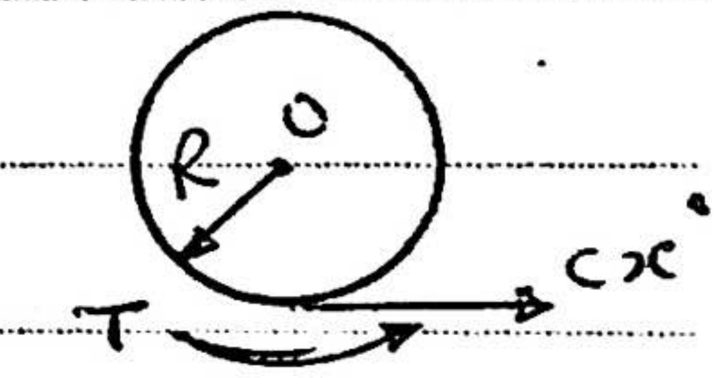
$$I\ddot{\theta} = -T - c\dot{x} \cdot R$$

But $x = R\theta$

$$I\ddot{\theta} = -k_T\theta - cR^2\dot{\theta}$$

$$I\ddot{\theta} + CR^2\dot{\theta} + k_T\theta = 0$$

$\therefore T = K_T\theta \Rightarrow$



$$K_T = \frac{T}{\theta} = \frac{\frac{J \cdot G \cdot \cancel{\theta}}{L}}{\cancel{\theta}}$$

$$\Rightarrow K_T = \frac{G \cdot J}{L}$$

$$\therefore I\ddot{\theta} + CR^2\dot{\theta} + \frac{G \cdot J}{L}\theta = 0$$

$$\ddot{\theta} + \frac{CR^2}{I}\dot{\theta} + \frac{G \cdot J}{LI}\theta = 0$$

$$\therefore 2\xi\omega_n = \frac{CR^2}{I} \Rightarrow \xi = \frac{CR^2}{2I\omega_n}$$

$$\omega_n^2 = \frac{G \cdot J}{LI}$$

Chapter Three

Forced Harmonic Vibration

Harmonic excitation is often encountered in engineering systems. It is commonly produced by the unbalance in rotating machinery. Although pure harmonic excitation is less likely to occur than periodic or other types of excitation, understanding the behavior of a system undergoing harmonic excitation is essential in order to comprehend how the system will respond to more general types of excitation. Harmonic excitation may be in the form of a force or displacement of some point in the system.

We will first consider a single DOF system with viscous damping, excited by a harmonic force $F_o Cos\,\omega t$, as shown in Figure 1. Its differential equation of motion is found from the free-body diagram.

$$m\ddot{x} + c\dot{x} + kx = F\,Cos\,\omega t \qquad ......(1)$$

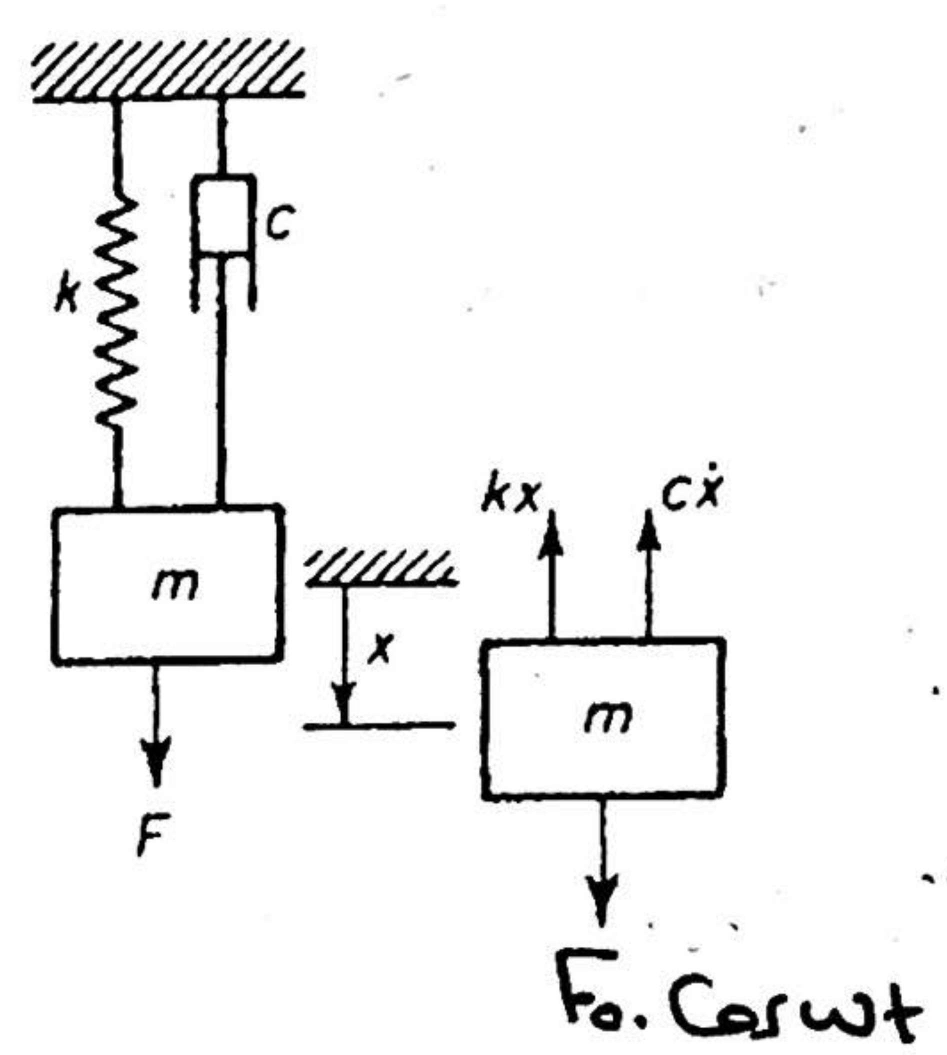


Figure 1. Viscously Damped System with Harmonic Excitation

الدالة المتممة

The solution to this equation consists of two parts, the complementary function, which is the solution of the homogeneous equation, and the particular integral. The complementary function. in this case, is a damped free vibration.

The particular solution to the preceding equation is a steady-state oscillation of the same frequency w as that of the excitation. We can assume the particular solution to be of the form :

$$x = X Cos(\omega t - \phi) \qquad ......(2)$$

The steps :-

$$m.\ddot{x} + C.\dot{x} + K.x = F_o.\cos\omega t \quad — ①$$

assume $x = X.\cos(\omega t - \phi)$

The Particular solution

$\dot{x} = -X.\omega.\sin \omega t - \phi$; $\ddot{x} = -X.\omega^2.\cos\omega t - \phi$ sub in ①

$$m(-X\omega^2.\cos\omega t - \phi) - C(X\omega.\sin\omega t - \phi) + K.(X\cos\omega t - \phi) = F_o.\cos\omega t$$

But $\cos\omega t - \phi = \cos\omega t.\cos\phi + \sin\omega t \sin\phi$

and $\sin\omega t - \phi = \sin\omega t\cos\phi - \sin\phi.\cos\omega t$ sub in ①

$$X\left[-m\omega^2.\cos\omega t.\cos\phi - m\omega^2\sin\omega t.\sin\phi\right] + X\left[-c\omega.\sin\omega t.\cos\phi + c\omega.\sin\phi.\cos\omega t\right] + X\left[K.\cos\omega t.\cos\phi + K.\sin\omega t.\sin\phi\right] = F_o\cos\omega t$$

$$X\left[-m\omega^2.\cos\omega t.\cos\phi + c\omega.\sin\phi.\cos\omega t + K.\cos\omega t.\cos\phi\right] = F_o.\cos\omega t$$

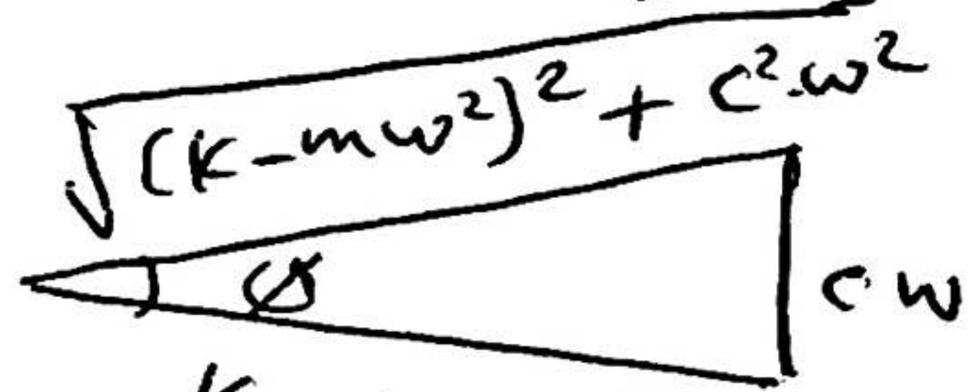
$$X\left[(K - m\omega^2).\cos\phi + c\omega.\sin\phi\right] = F_o \quad — *$$

so $X\left[-m\omega^2\sin\omega t.\sin\phi - c\omega.\sin\omega t.\cos\phi + K.\sin\omega t\sin\phi\right] = 0$

$\therefore X\left[(K - m\omega^2)\sin\phi - c\omega.\cos\phi\right] = 0 \quad — (**)$

$\therefore (K - m\omega^2).\sin\phi = c\omega.\cos\phi \Rightarrow \tan\phi = \dfrac{c.\omega}{(K - m\omega^2)}$

$\therefore \sin\phi = \dfrac{c.\omega}{\sqrt{(K - m\omega^2)^2 + c^2\omega^2}}$; $\cos\phi = \dfrac{K - m\omega^2}{\sqrt{(K - m\omega^2) + c^2\omega^2}}$



$\therefore$ sub in (*)

$$X\left(\frac{((K - m\omega^2)^2 + c^2\omega^2)}{\sqrt{(K - m\omega^2)^2 + C^2\omega^2}}\right) = F_o$$

$\therefore X = \dfrac{F_o}{\sqrt{(K - m\omega^2)^2 + c^2\omega^2}}$ divided on K

$$\boxed{X = \frac{F_o/K}{\sqrt{\left(1 - \frac{m\omega^2}{K}\right)^2 + \left(\frac{c.\omega}{K}\right)^2}}}$$

; ALSO $\boxed{\tan\phi = \dfrac{\frac{c.\omega}{K}}{\left(1 - \frac{m\omega^2}{K}\right)}}$

But $w_n = \sqrt{\dfrac{K}{m}}$ and $C_c = 2m \cdot w_n$; $\zeta = \dfrac{c}{c_c}$ (*)

$$\frac{c \cdot w}{K} = \frac{C_c}{C_c} \cdot \frac{c \cdot w}{K} = \zeta \cdot \frac{2\overset{1/w_n^2}{m} \cdot w_n \cdot w}{K} = 2\zeta \cdot \frac{w_n \cdot w}{w_n^2} = 2\zeta \cdot \frac{w}{w_n}$$

$\therefore \dfrac{c \cdot w}{K} = 2\zeta \cdot \dfrac{w}{w_n} \longrightarrow$ (*) sub

$$\boxed{X = \frac{F_o/K}{\sqrt{\left(1 - \frac{w^2}{w_n^2}\right)^2 + \left(2\zeta \cdot \frac{w}{w_n}\right)^2}}}$$

Note/ $\dfrac{F_o}{K} = \delta_{static}$ deflection

$$\therefore \boxed{\tan\phi = \frac{2\zeta \cdot \frac{w}{w_n}}{1 - \left(\frac{w}{w_n}\right)^2}}$$

The total response can be write $x(t) = x|_p + x|_h$

$\therefore x(t) = X \cdot \cos(wt - \phi) + X_o \cdot e^{-\zeta w_n t} \cos(w_d t - \phi_o)$

$w_d = \sqrt{1 - \zeta^2} \cdot w_n$

$$\therefore \boxed{x(t) = \frac{F_o/K \cdot \cos(wt - \phi)}{\sqrt{\left(1 - \frac{w^2}{w_n^2}\right)^2 + \left(2\zeta \cdot \frac{w}{w_n}\right)^2}} + X_o \cdot e^{-\zeta w_n t} \cdot \cos\left(\sqrt{1 - \zeta^2} \cdot w_n - \phi_o\right)}$$

or

$x(t) = X \sin(wt - \phi) + X_o \cdot e^{-\zeta w_n t} \cdot (\sin w_d t - \phi_o)$

$$\therefore \boxed{x(t) = \frac{F_o/K \cdot \sin(wt - \phi)}{\sqrt{\left(1 - \frac{w^2}{w_n^2}\right)^2 + \left(2\zeta \cdot \frac{w}{w_n}\right)^2}} + X_o \cdot e^{-\zeta w_n t} \cdot \sin\left(\sqrt{1 - \zeta^2} \cdot w_n - \phi\right)}$$

لنفس إذ في المعادلة أعلاه هو عندما $\zeta < 1$
ملاحظة مهمة / ممكن أن نلغي الحد الثاني إعتماداً على قيمة (ζ) معامل التخميد ١ - ويمكن أن نلغي عندما $\zeta = 1$ و $\zeta > 1$

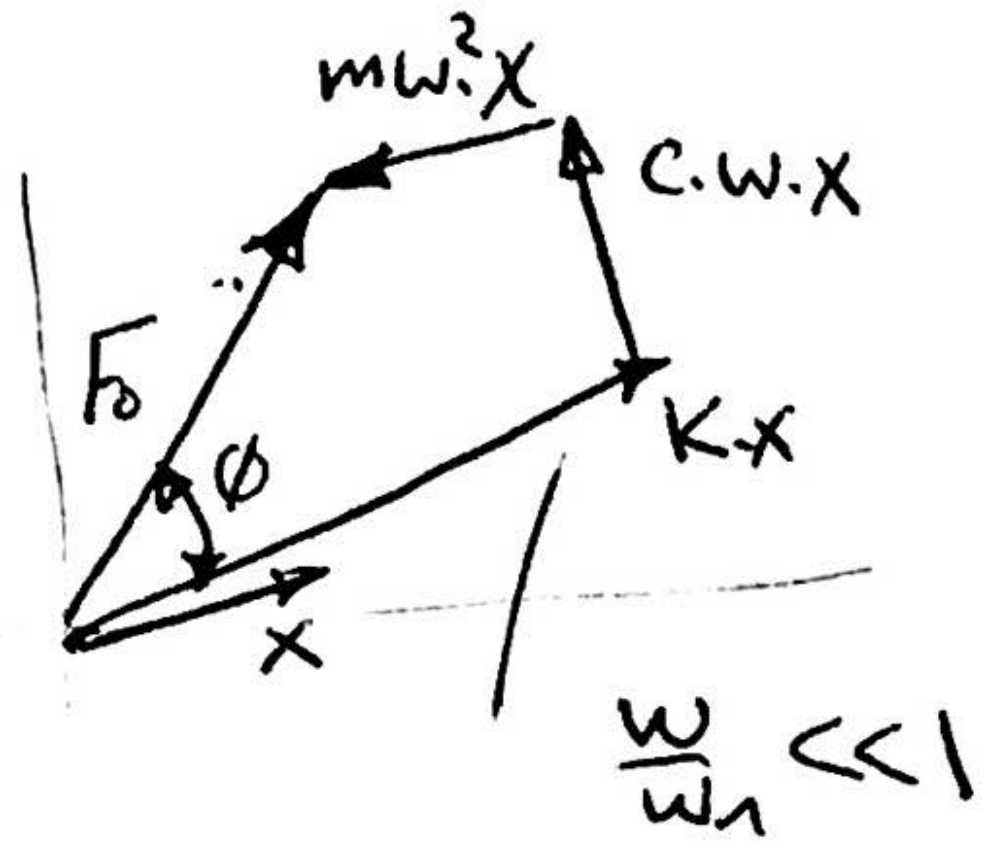
Finally :- we can ~~be~~ describe the vibration system, by studying the ratio $\frac{w}{w_n}$ into three cases

*Case ① where $w/w_n \ll 1$ we get the diagram as shown :-

Note/ ① ϕ = its very small or $\phi < 90°$

② $m w^2 . X$ = its very small

③ $C . w . X$ = its very small



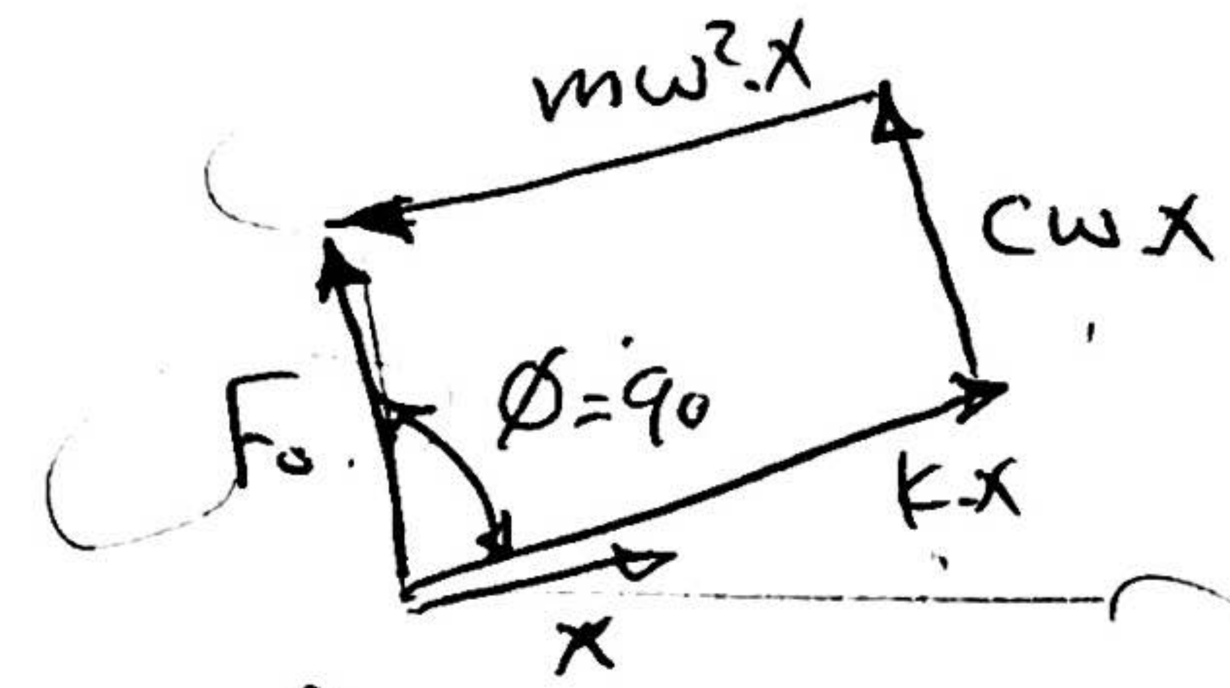
* Case ②

① $\phi = 90°$

② $m w^2 . X = K . X$

③ $F_o = C . w . X$

Note/ where $\frac{w}{w_n} = 1$ ∴ $\left(X = \frac{F_o}{C . w_n} = \frac{F_o}{2\xi . K} \right)$ $\frac{w}{w_n} = 1$



* Case ③

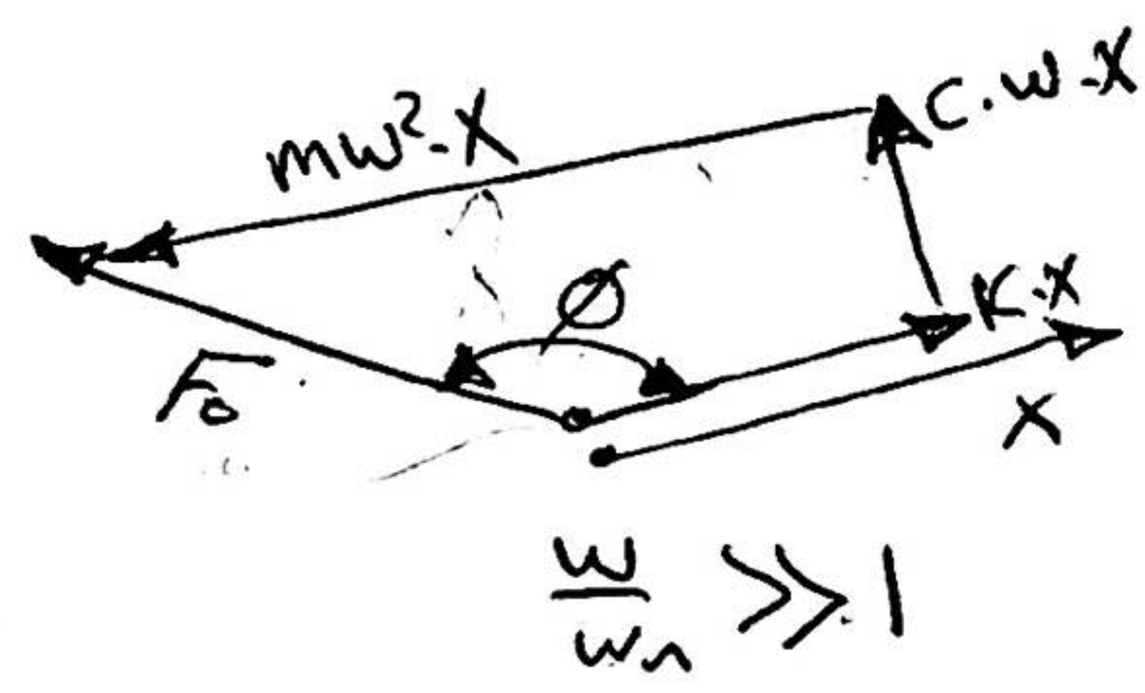
① $\phi = 180$ or $\phi > 90$

②

* $F_o > m w^2 . X$ قوة القصور

* $m w^2 . X > K . X$ قوة الإرجاع

③ $F_o > C w . X$ قوة التخميد



السعة الرنينية

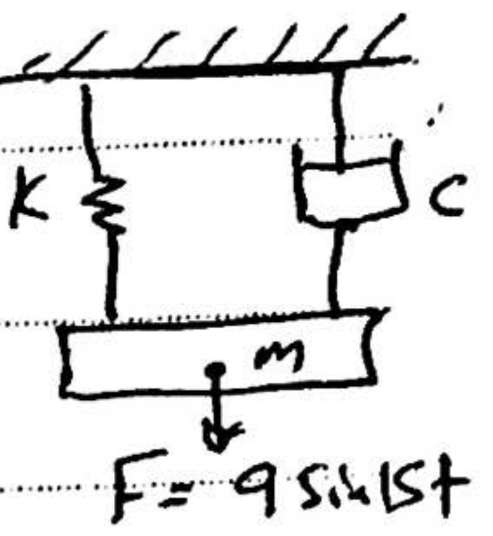
Note// ① $\left[X = \frac{F_o}{K . (2\xi)} \right]$ The resonant amplitude

② resonant frequancy $\left(\frac{W}{W_n} \right) = ?$

③ The peak amplitude $\left(\frac{w}{w_n} \right)_p = \sqrt{1 - 2\xi^2}$

④ the angle phase of resonant $\tan\phi = 2\xi . \frac{w}{w_n} / (1 - (\frac{w}{w_n})^2$

Example: A system of mass-spring-damper have $m = 5\,kg$; $k = 4.5\ kN/m$; $c = 300\ N.s/m$, $F = 9\sin 15t\ kN$, the initial conditions are $\dot{x}(0) = 0$ and $x(0) = 5\ cm$. Find the equation of motion of this system.



solution:

$$\omega_n = (k/m)^{1/2} = (4.5*10^3/5)^{1/2} = 30\ rad/sec.$$

$$\xi = C/C_c = 300/(2*5*30) = 1 \qquad \text{(Critical damping)}$$

$$\therefore\ x_c(t) = (A + Bt)\,e^{-\omega_n t} = (A + Bt)\,e^{-30t}$$

$$x_p(t) = \frac{F_o}{k} \frac{\sin(15t - \phi)}{\sqrt{[1 - (\frac{\omega}{\omega_n})^2]^2 + (2\xi\frac{\omega}{\omega_n})^2}} = X \sin(\omega t - \phi)$$

where $\tan\phi = \dfrac{2\xi\frac{\omega}{\omega_n}}{1 - (\frac{\omega}{\omega_n})^2}$

$$X = \frac{9000/4500}{\sqrt{[1 - (\frac{15}{30})^2]^2 + (2*1*\frac{15}{30})^2}} = 1.6\,m.$$

and $\phi = \tan^{-1}\left(\dfrac{2*1*\frac{15}{30}}{1 - (\frac{15}{30})^2}\right) = 53.13^\circ = 0.93\ rad$

$$x_p(t) = 1.6\sin(15t - 0.93)$$

$$\boxed{x(t) = (A + Bt)\,e^{-30t} + 1.6\sin(15t - 0.93)}$$

at $t = 0 \Rightarrow x(0) = 0.05 = A - 1.28 \Rightarrow A = 1.23$

$$\dot{x}(0) = 0 = -30A + B + 14.42$$

$$\Rightarrow B = 22.48$$

$$\therefore\ x(t) = (1.23 + 22.48t)\,e^{-30t} + 1.6\sin(15t - 0.93).$$

وسط معين

Example: A mechine of weight 2.15 kg in a discase medium determine the damping coffecient when a harmonic excitation force of 27.5 sin ωt. Results in resonace amplitude of 1.25 cm with a periade of 0.2 sec. $\omega = \frac{2\pi}{\tau} = \omega_n$

Solution:

at resonance $\Rightarrow \omega_n = \omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.2} = 10\pi$ rad/sec.

$\therefore X = \frac{F_o/K}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + [2\xi\frac{\omega}{\omega_n}]^2}} = \frac{F_o}{K} \cdot \frac{1}{2\xi} = \frac{27.5}{2\xi K}$

$\therefore \frac{1.25}{100} = \frac{27.5}{2\xi K} \Rightarrow \xi K = \frac{27.5 * 100}{2.5} = 1100\ N/m$

$\xi K = \frac{C}{C_c} K = \frac{C}{2m\omega_n} k = \frac{C \cdot \omega_n}{2} = \frac{C}{2} * 10\pi = 5\pi C$

$\Rightarrow \therefore C = 1100/5\pi = 70\ N.S/m.$

Example: A weight at attached to a spring of stiffness (6 N/cm) has a descass damping device when the weight was displase and relased, the periode of vibration was found to be 1.8 sec & the ratio consective amplitude was 4.2 to 1. Find the amplitude of the phase when a force F = 10 cos 3t acts on the system.

Solution:

$F = F_o \sin\omega t \Rightarrow F_o = 10,\ \omega = 3,\ \omega_n = \frac{2\pi}{\tau} = \frac{2\pi}{1.8} = 3.49$ rad/sec

$f_r = \frac{\omega}{\omega_n} = \frac{3 * 1.8}{2\pi} = 0.859$

$\ln\frac{x_1}{x_2} = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \Rightarrow \xi =$

$x(t) = \frac{F_o/K}{\sqrt{[1-f_r^2] + [2\xi f_r]^2}} = \frac{10 * 100/6}{\sqrt{[1-0.859^2] + [2\xi * 0.859]^2}} =$

$\phi = \tan^{-1}\frac{2\xi f_r}{1-f_r^2}$

Example: Show that for the damping spring - mass system. the peak amplitude occurs at a frequency ratio given by the expression: $\frac{\omega}{\omega_n}\Big|_p = \sqrt{1-2\xi^2}$

solution:

$$X = \frac{\frac{F_o}{k}}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + (2\xi\frac{\omega}{\omega_n})^2}}$$

$$\frac{dX}{d(\frac{\omega}{\omega_n})^2} = 0 \Rightarrow$$

$$\Rightarrow 0 = \frac{-\frac{F_o}{k}[\frac{1}{2}(2(1-(\frac{\omega}{\omega_n})^2) * -2(\frac{\omega}{\omega_n}) + 2(2\xi\frac{\omega}{\omega_n}) * 2\xi] * ((1-(\frac{\omega}{\omega_n})^2)^2 + (2\xi\frac{\omega}{\omega_n})^2)^{-1/2}}{[(1-(\frac{\omega}{\omega_n})^2)^2 + (2\xi\frac{\omega}{\omega_n})^2]}$$

$$\Rightarrow 0 = \frac{-\frac{F_o}{k}[\frac{1}{2}(2(1-(\frac{\omega}{\omega_n})^2) * -2(\frac{\omega}{\omega_n}) + 8\xi^2(\frac{\omega}{\omega_n})}{[(1-(\frac{\omega}{\omega_n})^2)^2 + (2\xi\frac{\omega}{\omega_n})^2]^{1/2} * [(1-(\frac{\omega}{\omega_n})^2)^2 + (2\xi\frac{\omega}{\omega_n})^2]}$$

$$\therefore -\frac{F_o}{k}[\frac{1}{2}(-4\frac{\omega}{\omega_n} + 4(\frac{\omega}{\omega_n})^3 + 8\xi^2\frac{\omega}{\omega_n} = 0$$

$$-4 + 4(\frac{\omega}{\omega_n})^2 + 8\xi^2 = 0$$

$$2\xi^2 - 1 + (\frac{\omega}{\omega_n})^2 = 0$$

$$\Rightarrow (\frac{\omega}{\omega_n})^2 = 1 - 2\xi^2$$

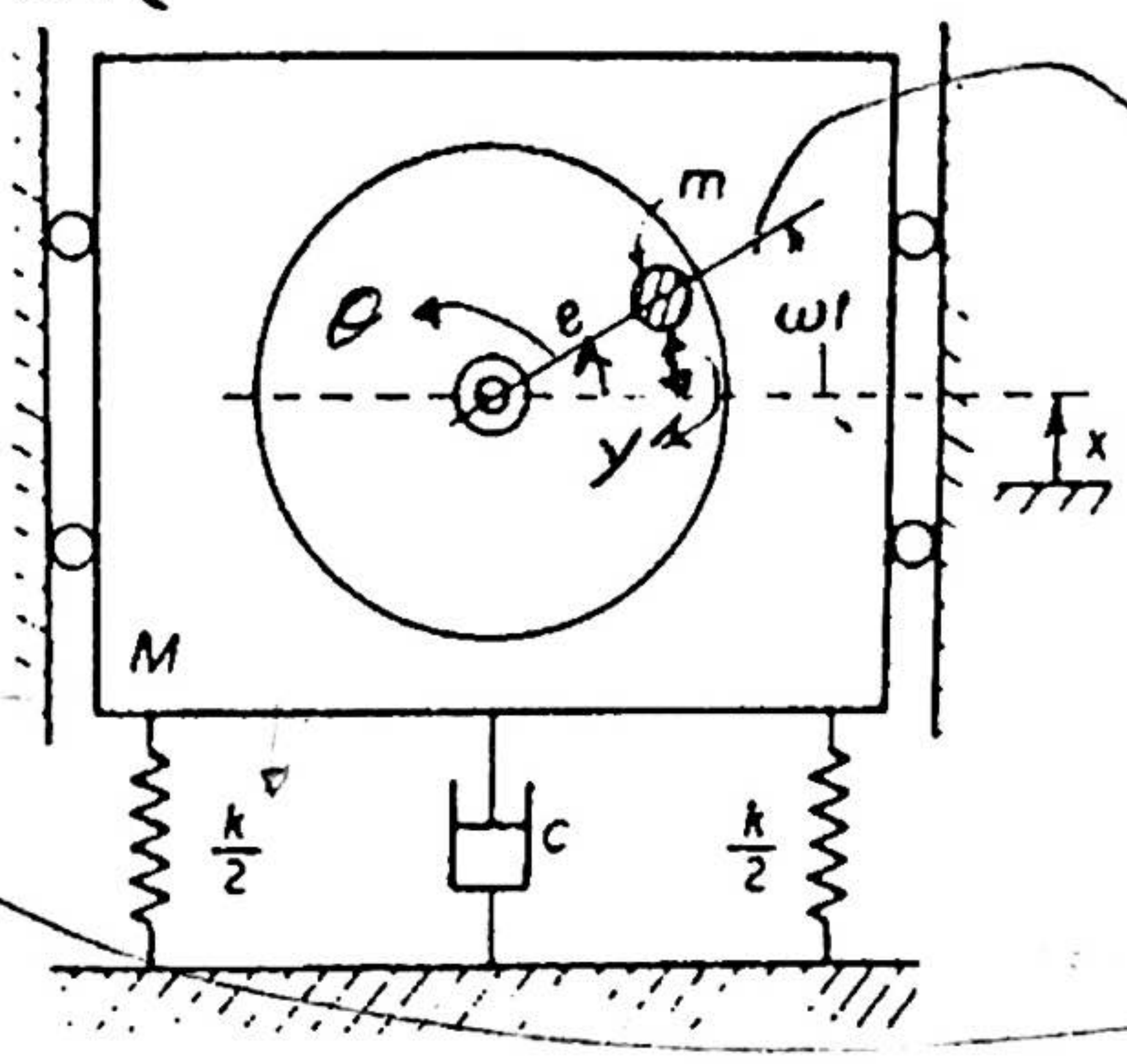
$$\therefore \frac{\omega}{\omega_n}\Big|_p = \sqrt{1-2\xi^2}.$$

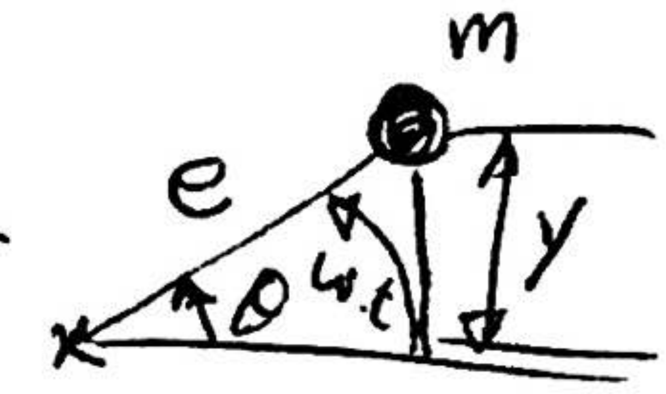
Rotating Unbalance

Unbalance in rotating machines is a common source of vibration excitation. We consider here a spring-mass system constrained to move in the vertical direction and excited by a rotating machine that is unbalanced, as shown in Figure 4. The unbalance is represented by an eccentric mass m with eccentricity e that is rotating with angular velocity w . By letting y be the displacement of the non rotating mass [illegible] m) from the [illegible] equilibrium position, the displacement of m is :

$y = e\sin\theta \Rightarrow$ But $\theta = \omega . t$

$y = e \sin \omega t$





$\sin\theta = \frac{y}{e}$

$\therefore y = e\sin\theta$

But $\theta = \omega . t$

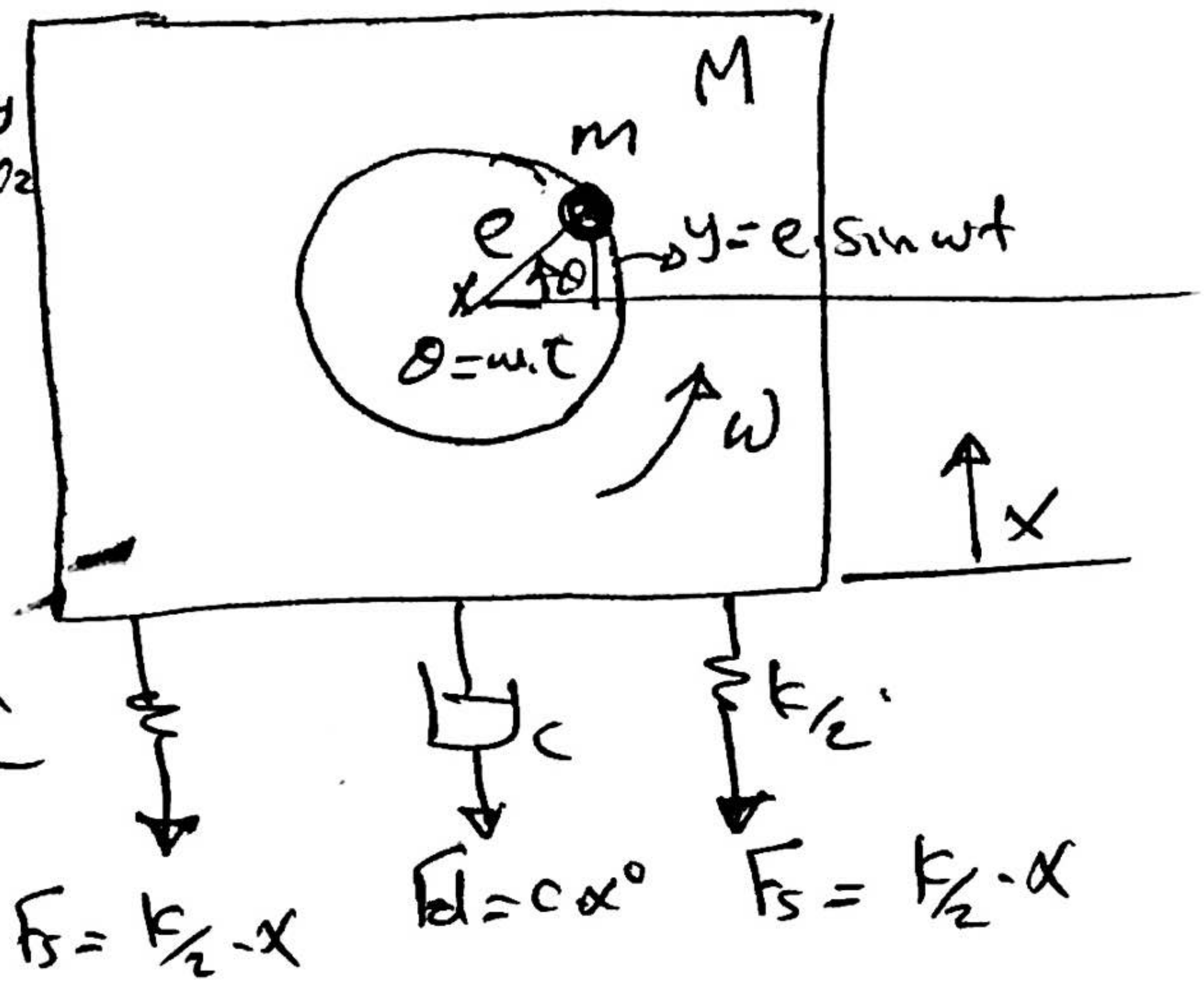
$\therefore y = e . \sin\omega t$

Figure 4. Harmonic Disturbing Force Resulting from Rotating Unbalance

(62)

$$\Sigma F = M.\ddot{x} + m.\frac{d^2y}{dt^2}$$

$$-K.\frac{x}{2} - K.\frac{x}{2} - C\dot{x} = M\ddot{x} + m.\frac{d^2y}{dt^2}$$



$\therefore$ $y = e.\sin\omega t$

$\dot{y} = e\omega.\cos\omega t$

$\ddot{y} = -e\omega^2.\sin\omega t$ sub. in equa

$$\therefore \quad -K.x - C.\dot{x} = M\ddot{x} + m.(-e.\omega^2.\sin\omega t)$$

$$\therefore \quad M.\ddot{x} + C\dot{x} + K.x = m.e.\omega^2 \sin\omega t$$

$$\Longleftrightarrow$$

$$M.\ddot{x} + C.\dot{x} + K.x = F_o . \sin\omega t$$

$$\therefore \quad F_o = me.\omega^2$$

$$\therefore \quad X = \frac{F_o/K}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta.\frac{\omega}{\omega_n}\right)^2}}$$

$$\therefore \quad X = \frac{me.\omega^2/K * \frac{M}{M}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta.\frac{\omega}{\omega_n}\right)^2}}$$

by multiplying $\frac{M}{M}$

$$\therefore \quad \frac{M}{m}.\frac{X}{e} = \frac{\left[\frac{\omega}{\omega_n}\right]^2}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\therefore \quad \tan\phi = 2\zeta.\omega/\omega_n \Big/ \left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)$$

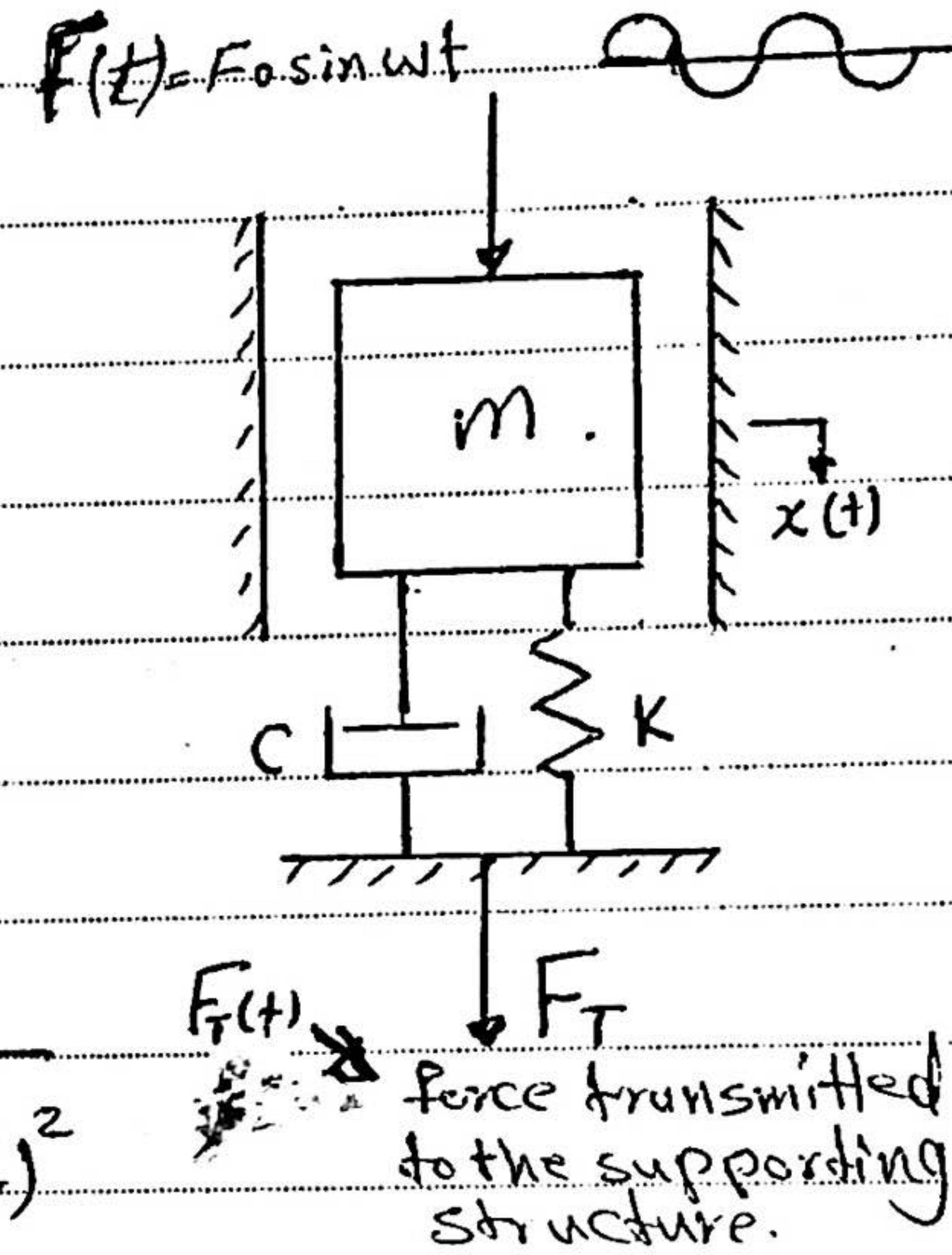
Vibration Isolation

Transmissibility (المستقلية)

Assuming that the forcing function is harmonic in nature, we shall consider two cases of vibration transmission—one in which force is transmitted to the supporting structure, and one in which the motion of the supporting structure is transmitted to the machine.

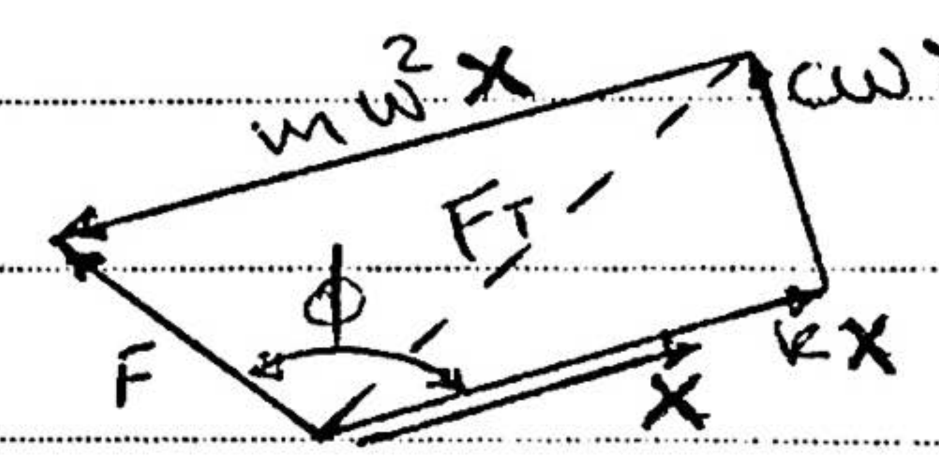
(a) Force Excitation

Consider the system show below, where $f(t)$ is the harmonic force acting on the system and $f_T(t)$ is the force transmitted to the supporting structure or base. The force transmitted through the spring and damper to the supporting structure is:



$$F_T = \sqrt{(kX)^2 + (c\omega X)^2} = kX\sqrt{1 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}$$

if $F = F_o \sin \omega t$, then:



$$X = \frac{\frac{F_o}{k}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}}$$

$$F_o = kX\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}$$

we define the Transmissibility, TR, as:

$$TR = \left|\frac{F_T}{F_o}\right| = \sqrt{\frac{1 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} = \left|\frac{X}{Y}\right|$$

ملاحظة مهمة / ان القوة المنتقلة الى الاساس = حاصل ضرب المستقلية × القوة المؤثرة

$$(F_o * TR) = F_T$$

* Remarks:-

If $\xi \to 0 \Rightarrow TR = \dfrac{1}{\left|\left(\frac{\omega}{\omega_n}\right)^2 - 1\right|}$ where $\frac{\omega}{\omega_n} > \sqrt{2}$

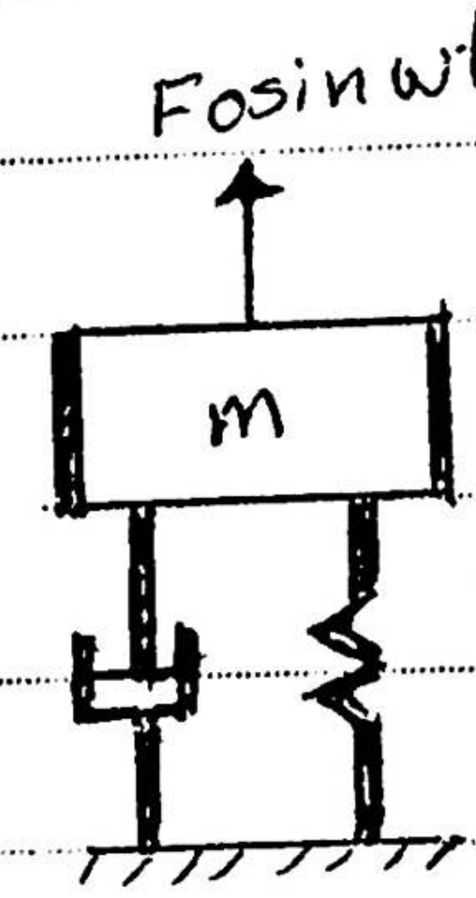
ناقلية مطلقة دائماً وهي الحوجم

Note: * In order to decrease ω_n we can either use a soft spring or increase the mass or both (since $\omega_n = \left(\frac{k}{m}\right)^{1/2}$).

* An undamped spring is superior to damped spring in reduceing transmissibility (i.e. when $\xi = 0$).

* The Percentage reduction of T.R: $R = (1 - T.R) * 100$

✓ Example: A machine has a disturbance frequency of (15 c.p.s) (cycle per second), it is mount on rubber bed with static deflection (0.75 cm). Find the percentag reduction of the transmitted force. Note: $\xi = 0$



solution:

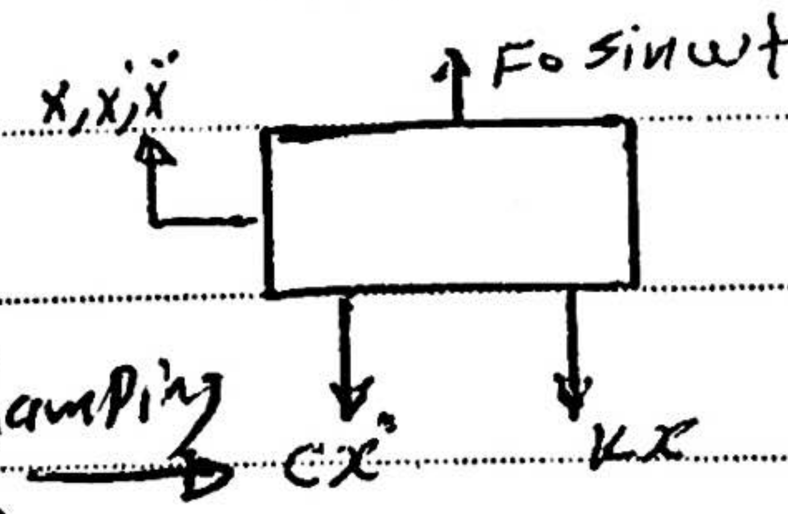
$\omega = 2\pi * 15 = 30\pi$ rad/sec

$\omega_n = \left(\frac{k}{m}\right)^{1/2} = \left(\frac{kg}{w}\right)^{1/2} = \left(\frac{kg}{k\delta}\right)^{1/2} = \left(\frac{9.81}{0.75 \times 10^{-2}}\right)^{1/2}$

$\Rightarrow \omega_n = (36.16)$ rad/sec

$f_r = \left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{30\pi}{36.16}\right)^2 = 6.8$

$\because \xi = 0$ (No damping) (static deflection)



$T.R = \dfrac{1}{|1 - f_r^2|} = \dfrac{1}{5.8} = 0.1725$

No damping ratio

$\therefore C = 0$ (F.B.D)

$\therefore \xi = 0$

% reduction in force transmitted is

$R = (1 - TR) * 100$

$= (1 - 0.1725) * 100$

$= 82.75$

ملاحظة مهم جداً/ عندما لا يوجد تخميد فان قيمة معامل التخميد ع = صفر
وهنا تشير ان المنظومة تحت عتبة التأثير الاستاتيكي للقوى

✓ Example: A machine of 200 kg mass is supported on springs of total stiffness 1500 kN/m and has an unbalanced rotating element which results in a sinusoidally fluctuating disturbing force of 400 N.
Amplitude at a speed of 3000 rpm. Assuming a damping factor of $\xi = 0.20$, determine (a) It's amplitude of motion due to the unbalance, (b) the transmissibility, (c) the transmitted force.

solution:

a. $\omega_n = (K/m)^{1/2} = (1500000/200)^{1/2} \doteq 36.8$ rad/sec

$\omega = 3000 * 2\pi/60 = 312.159$ rad/sec.

$f_r = \frac{\omega}{\omega_n} = 312.159/36.8 = 3.6276$.

$$X = \frac{F_o/k}{\sqrt{(1-(f_r)^2)^2 + (2\xi f_r)^2}} = \frac{400/1500000}{\sqrt{(1-(3.6276)^2)^2 + (2*0.2*3.6276)^2}}$$

$= 0.02177$ mm

b. $$TR = \frac{F_T}{F_o} = \frac{\sqrt{1+(2\xi f_r)^2}}{\sqrt{(1-f_r^2)^2 + (2\xi f_r)^2}}$$

$$= \frac{\sqrt{1+(2*0.2*3.6276)^2}}{\sqrt{(1-(3.6276)^2)^2 + (2*0.2*3.6276)^2}}$$

$= 0.143907$

c. The transmitted force $F_T = TR * F_o$

$= 0.143907 * 400$

$= 57.56$ N.

Example: A machine of 700 kg mass is supported on springs of total stiffness 700 KN/m and has an unbalanced rotating element which results in a disturbing force of 350 N at a speed of 3000 rev/min. Assuming a damping factor of $\xi = 0.2$, determine:

(a) It's amplitude of motion due to the unbalance.

(b) The transmissibility.

(c) The transmitted force.

solution:

a. $\omega_n = (K/m)^{1/2} = (700 \ast 10^3/100)^{1/2} = 83.66$ rad/sec.

$\omega = 3000 \ast \frac{2\pi}{60} = 314.16$ rad/sec.

$$X = \frac{\left(\frac{F_o}{K}\right)}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{(350/700 \ast 10^3)}{\sqrt{\left(1-\left(\frac{314.16}{83.66}\right)^2\right)^2 + \left(2 \ast 0.2 \ast \frac{314.16}{83.66}\right)^2}} = 0.0379 \text{ mm}$$

$$TR = \sqrt{\frac{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}{\left(1+\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\left(\frac{\omega}{\omega_n}\right)\right)^2}}$$

$$= \sqrt{\frac{1+\left(2 \ast 0.2 \ast \frac{314.16}{83.66}\right)^2}{\left(1-\left(\frac{314.16}{83.66}\right)^2\right)^2 + \left(2 \ast 0.2 \ast \frac{314.16}{83.66}\right)^2}} = 0.137$$

c. $TR = \frac{F_T}{F_o} \Rightarrow F_T = TR \cdot F_o$

$= 0.137 \ast 350$

$= 47.89$ N.

Example: Consider the machine of mass 1000 kg is supported on a vertical flexible mounting, modelled as a single degree-of-freedom system. The mounting has a total stiffness 50 kNm^{-1} and viscous damping is 5 $kNm^{-1}s$. Calculate the response amplitude, the phase angle, and the force transmitted to the foundations when the driving frequency is (a) 20 HZ and (b) 2 HZ. The amplitude of the applied force is 2500 N.

solution:

a. $m = 1000\ kg$, $F_o = 2500\ N$, $c = 5\ kNm^{-1}s$

$\omega = 20 HZ = 2\pi * 20 = 125.7\ rad\ s^{-1}$, $k = 50\ kNm^{-1}$.

$\therefore \omega_n = 7.07\ rad s^{-1}$, and $r = \frac{\omega}{\omega_n} = 17.8$

$$\xi = c/2\sqrt{km} = 5*10^3/2\sqrt{50*10^3 * 1000} = 0.354$$

$$X = \frac{F_o/k}{\sqrt{(1-r^2)^2+(2\xi r)^2}} = \frac{2500/(50*10^3)}{\sqrt{(1-17.8^2)^2+(2*0.354*17.8)^2}}$$

$$= \frac{0.05}{\sqrt{(-315.84)^2+12.6^2}} = \frac{0.05}{316.1} = 1.58*10^{-4}\ m.$$

So the response amplitude is still very small at less than 0.16 mm.

$$\Rightarrow \phi = \tan^{-1}\frac{2\xi r}{1-r^2} = \tan^{-1}\frac{2*0.354*17.8}{1-17.8^2}$$

$$= \tan^{-1}\frac{12.6}{-315.84} = \tan^{-1} -0.04 = 177.7^\circ$$

The amplitude of the force transmitted to the foundations:

$$F_T = F_o T_R = F_o \left[\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2} \right]^{1/2}$$

$$= 2500 * \left[\frac{1+(2*0.354*17.8)^2}{(1-17.8^2)^2+(2*0.354*17.8)^2} \right]^{1/2} = 100\,N$$

b. $\omega = 2\,HZ = 2\pi * 2 = 12.57\ rad\,s^{-1}$, and $r = 1.78$

$$X = \frac{F_o/k}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

$$= \frac{2500/(50*10^3)}{\sqrt{(1-1.78^2)^2+(2*0.354*1.78)^2}} = \frac{0.05}{6.297} = 7.94*10^{-3}\,m$$

$$\tan\phi = \frac{2\xi r}{1-r^2} = \frac{2*0.354*1.78}{1-1.78^2} = \frac{1.26}{-2.16} = -0.58$$

$$\therefore \phi = \tan^{-1} -0.58 = 149.7^\circ$$

The amplitude of the force transmitted to the foundations:

$$P_o = F_o T = F_o \left[\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2} \right]^{1/2}$$

$$= 2500 * \left[\frac{1+(2*0.354*1.78)^2}{(1-1.78^2)^2+(2*0.354*1.78)^2} \right]^{1/2}$$

$$= 2500 * \left[\frac{1+1.26^2}{(-2.16)^2+(1.26)^2} \right]^{1/2} = 1608.3\,N$$

The damping has reduced the response amplitude at the lower speed, but in both cases, because r is greater than $\sqrt{2}$, the force transmitted has increased with the introduction of damping.

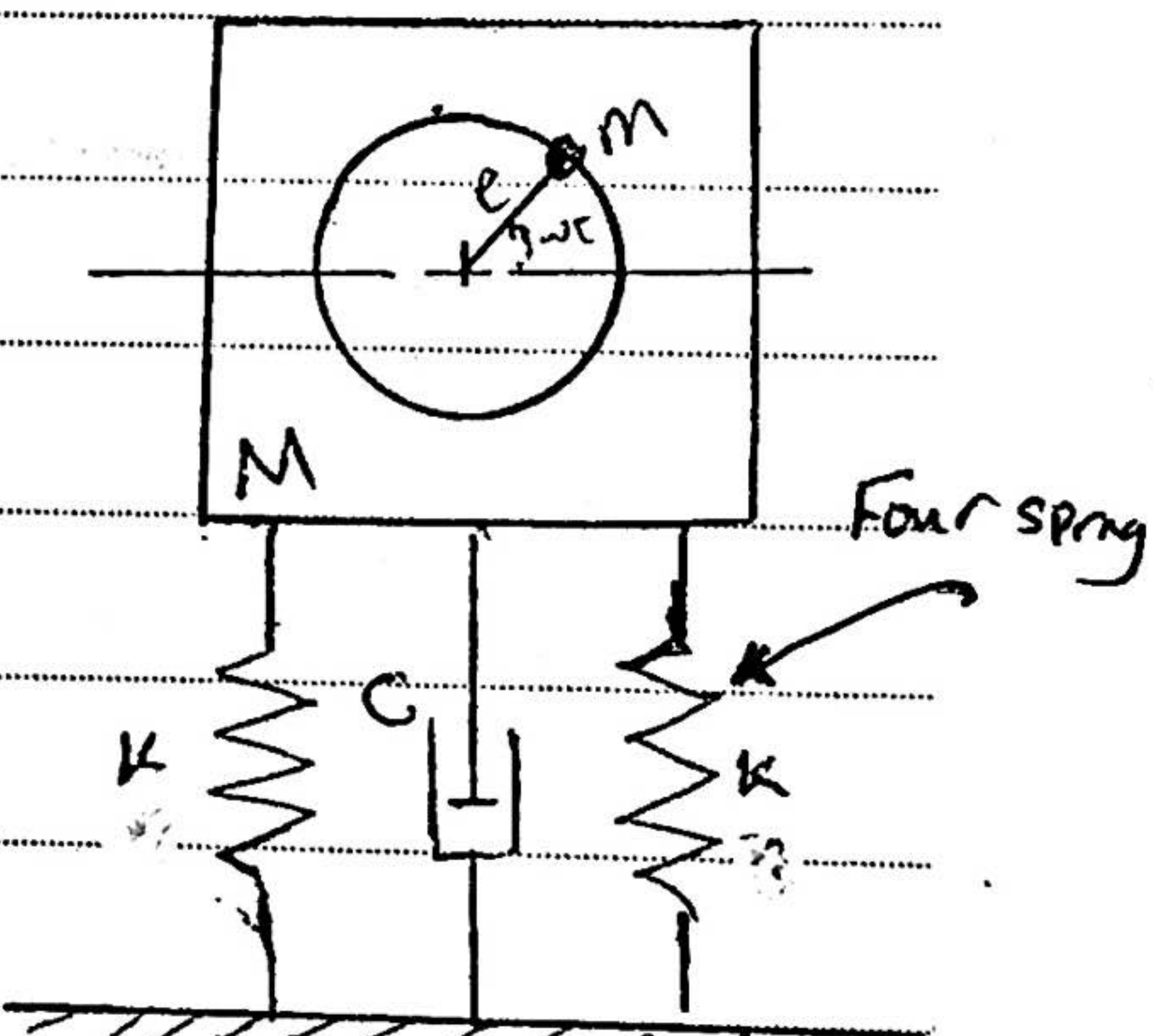
Example: A 40 kg motor is similar to the system shown in figure below. It is supported by four springs each of stiffness 250 Nm^{-1}. The rotor is unbalanced such that the unbalance effect is equivalent to a mass of 5 kg located 50 mm from the axis of rotation. Find the amplitude of vibration and the force transmitted to the foundation when the speed of the motor is (a) 1000 rev/min, (b) 60 rev/min. The damping ratio $\xi = 0.15$.

solution:

$M = 40\,kg$, $k = 4 * 250 = 1000\ Nm^{-1}$

$m = 5\,kg$, $e = 50\,mm = 0.05\,m$

$\xi = 0.15$



*Speed of motor = 1000 $rev.m^{-1}$

$\Rightarrow \omega = \frac{1000 * 2\pi}{60} = 104.7\ rad.s^{-1}$

$\omega_n = (k/m)^{1/2} = (1000/40)^{1/2} = 5\ rad.s^{-1}$

$\therefore r = \omega/\omega_n = 104.7/5 = 20.94$

$$X = \frac{mr^2e/M}{\sqrt{(1-r^2)^2+(2\xi r)^2}} = \frac{5 * 20.94^2 * 0.05/40}{\sqrt{(1-20.94^2)^2+(2*0.15*20.94)^2}}$$

$$= \frac{2.74}{\sqrt{191391.9+39.5}} = \frac{2.74}{437.5} = 6.3 * 10^{-3}\ m$$

$$\therefore F_T = F_0\left[\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}\right]^{1/2} = me\omega^2\left[\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}\right]^{1/2}$$

$$\therefore F_T = 5 * 0.05 * (104.7)^2\left[\frac{1+(2*0.15*20.94)^2}{(1-20.94^2)^2+(2*0.15*20.94)^2}\right]^{1/2}$$

$$= 2740.5\left[\frac{40.46}{191431.4}\right]^{1/2} = 39.84\ N.$$

* speed of motor $= 60\ rev\ min^{-1} = \frac{60*2\pi}{60} = 6.28\ rad\ s^{-1}$

and $r = \frac{\omega}{\omega_n} = \frac{6.28}{5} = 1.26$

$$X = \frac{mr^2e/M}{\sqrt{(1-r^2)^2+(2\xi r)^2}} = \frac{5*1.26^2*0.05/40}{\sqrt{(1-1.26^2)^2+(2*0.15*1.26)^2}}$$

$$= \frac{0.00992}{\sqrt{0.345+0.143}} = \frac{0.00992}{0.698} = 0.014\ m$$

$$\therefore F_T = 5 * 0.05 * 6.28^2\left[\frac{1+(2*0.15*1.26)^2}{(1-1.26^2)^2+(2*0.15*1.26)^2}\right]^{1/2}$$

$$= 9.86\left[\frac{1.143}{0.488}\right]^{1/2} = 15.1\ N$$

Example: An electric motor of mass 68 kg is mounted on an isolator block of mass 1200 kg and the natural frequency of the total assembly is 160 c.p.m with a damping factor of $\xi = 0.1$. If there is an unbalance in the motor that results in a harmonic force of $F = 100 \sin 31.4t$, determine the amplitude of vibration of the block and the force transmitted to the floor.

Solution:

$$\omega_n = \left(160 * \frac{2\pi}{60}\right) = 16.755 \text{ rad/sec.}$$

$$m_{total} = 1200 + 68 = 1268 \text{ kg}$$

$$\omega_n = (k/\underbrace{m}_{=1268})^{1/2} \Rightarrow k = \omega_n^2 \cdot m = (16.755)^2 * 1268$$

$$= 356 \text{ kN/m}.$$

$$X = \frac{F_o/k}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$\Rightarrow X = \frac{100/(356*10^3)}{\sqrt{\left(1-\left(\frac{31.4}{16.755}\right)^2\right)^2 + \left(2*0.1*\frac{31.4}{16.755}\right)^2}} = 0.1106 \text{ mm}$$

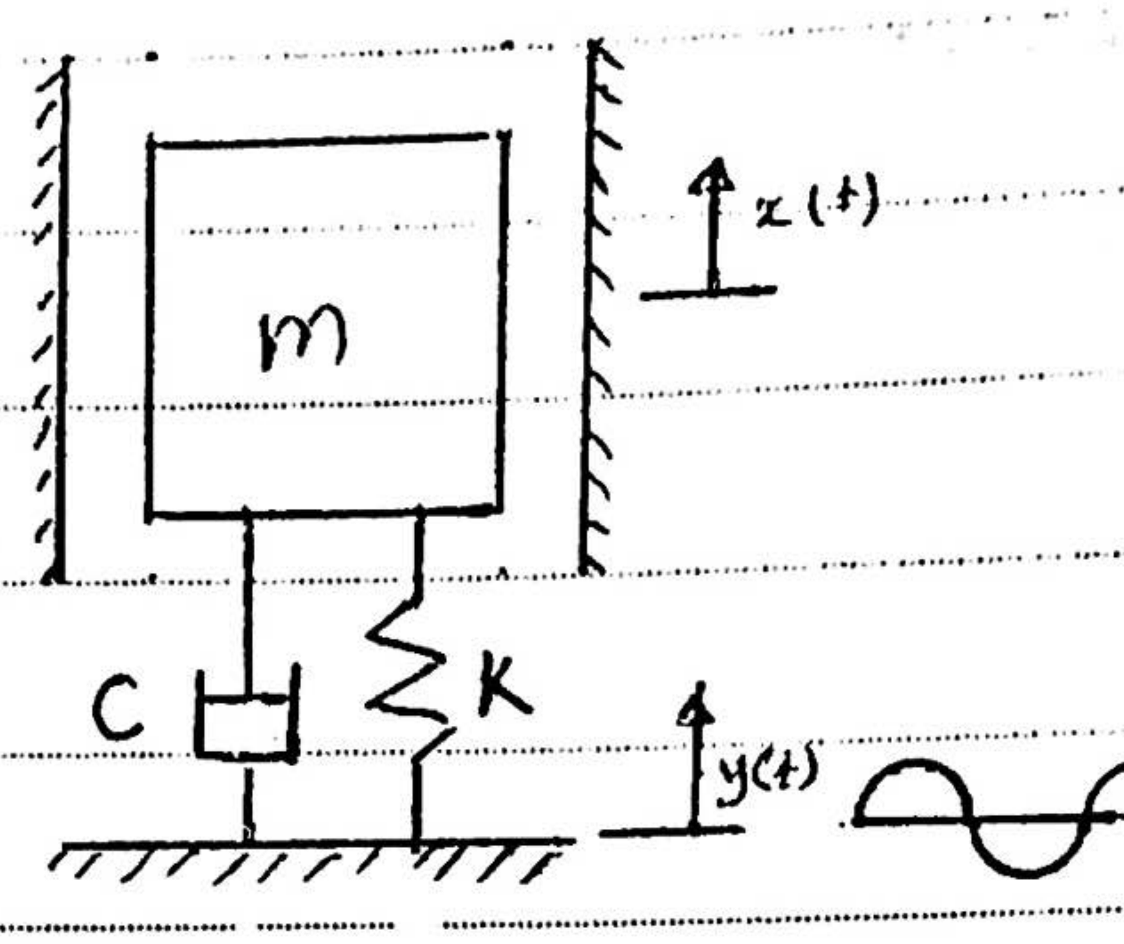
$$F_T = F_o * \sqrt{\frac{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}$$

$$= 100 * \sqrt{\frac{1 + \left(2*0.1*\frac{31.4}{16.755}\right)^2}{\left(1-\left(\frac{31.4}{16.755}\right)^2\right)^2 + \left(2*0.1*\frac{31.4}{16.755}\right)^2}}$$

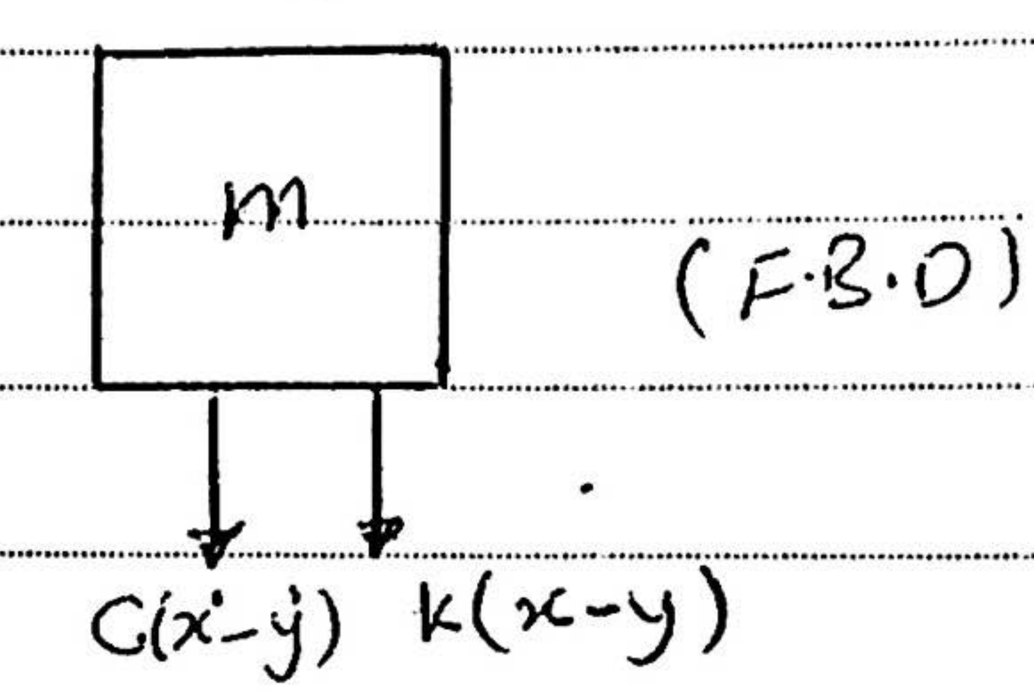
$$= 42 \text{ N}.$$

(b) Motion Excitation:

The system that illustrates motion excitation is shown. The motion of the dynamic system is represented by the variable x and the harmonic displacement of the supporting base is represented by the variable y. The equation that describes the dynamics of the system is:



From figure show that $x > y$



$$m\ddot{x}\uparrow = \sum F\uparrow$$

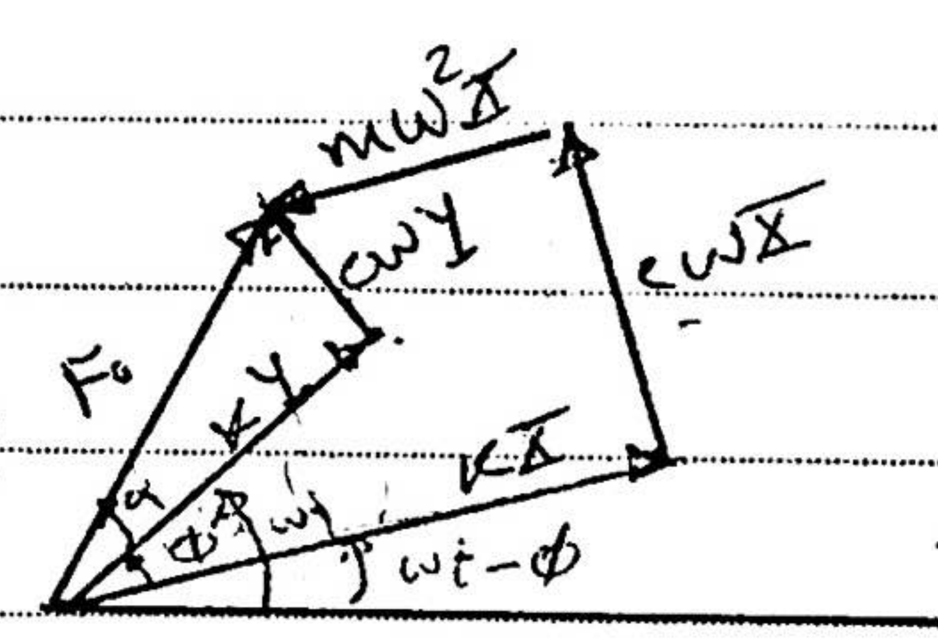
$$= -k(x-y) - c(\dot{x}-\dot{y})$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

let $y = Y \sin\omega t$

$$x = X \sin(\omega t - \phi)$$

$$\therefore\ m\omega^2 X \sin(\omega t-\phi) + c\omega X \cos(\omega t - \phi)$$
$$+ kX \sin(\omega t - \phi) = c\omega Y \cos\omega t + kY \sin\omega t$$



$$\sqrt{(kX - m\omega^2 X)^2 + (c\omega X)^2} = \sqrt{(kY)^2 + (c\omega Y)^2}$$

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \sqrt{\frac{1 + (2\zeta \frac{\omega}{\omega_n})^2}{(1 - (\frac{\omega}{\omega_n})^2)^2 + (2\zeta\frac{\omega}{\omega_n})^2}} = TR$$

$$\boxed{\tan\alpha = \frac{c\omega Y}{kY} = \frac{c\omega}{k} = \frac{2\zeta \cdot \omega/\omega_n}{1}}$$

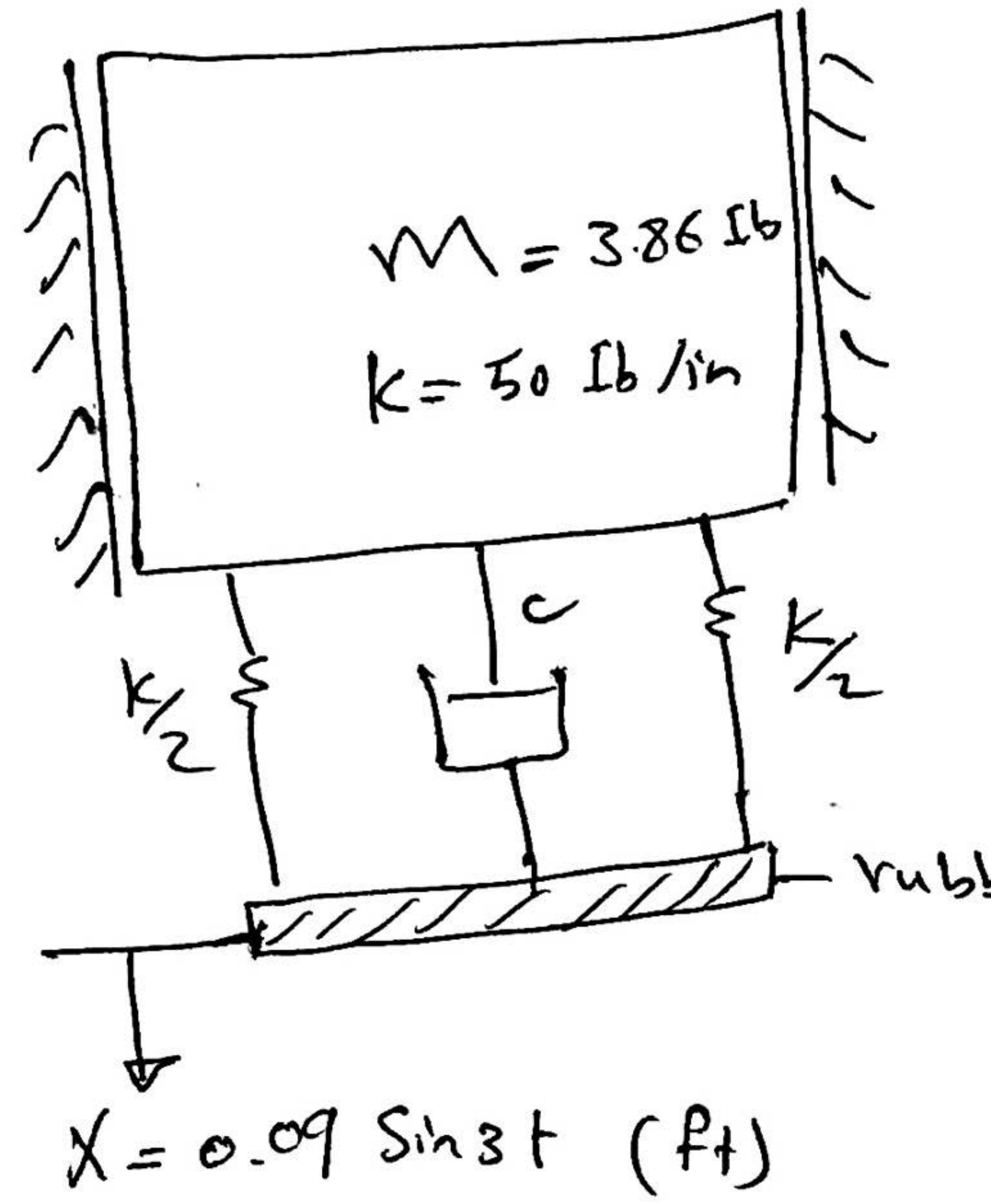
$$\boxed{\tan\phi = 2\zeta(\frac{\omega}{\omega_n})^3 \Big/ (1 - (\frac{\omega}{\omega_n})^2 + (2\zeta\frac{\omega}{\omega_n})^2}$$

$$x(t) = x_p(t) + x_h(t)$$

$$x(t) = X \cdot \sin(\omega t - \phi) + X_0 \cdot e^{-\zeta\omega_n t} \cdot \sin(\omega_d t - \phi)$$

Example 1:- For the system shown in Fig, Find
1. The amplitude of the mass
2. The max acceleration.

Assume that $\zeta = 0$ (No damping affects the rubber)



Example 2:- For the same mechanical vibration system up, Find
1. amplitude of the mass and angle phases when $\zeta = 0.2$
2. The max acceleration.
3. Compare between two amplitudes.
4. TR. (Transmissibility)

Answer Example 1

Sol:- No damping → $\xi = 0$ at rubber supporting

① $X = ?$

② $\ddot{X} = ?$

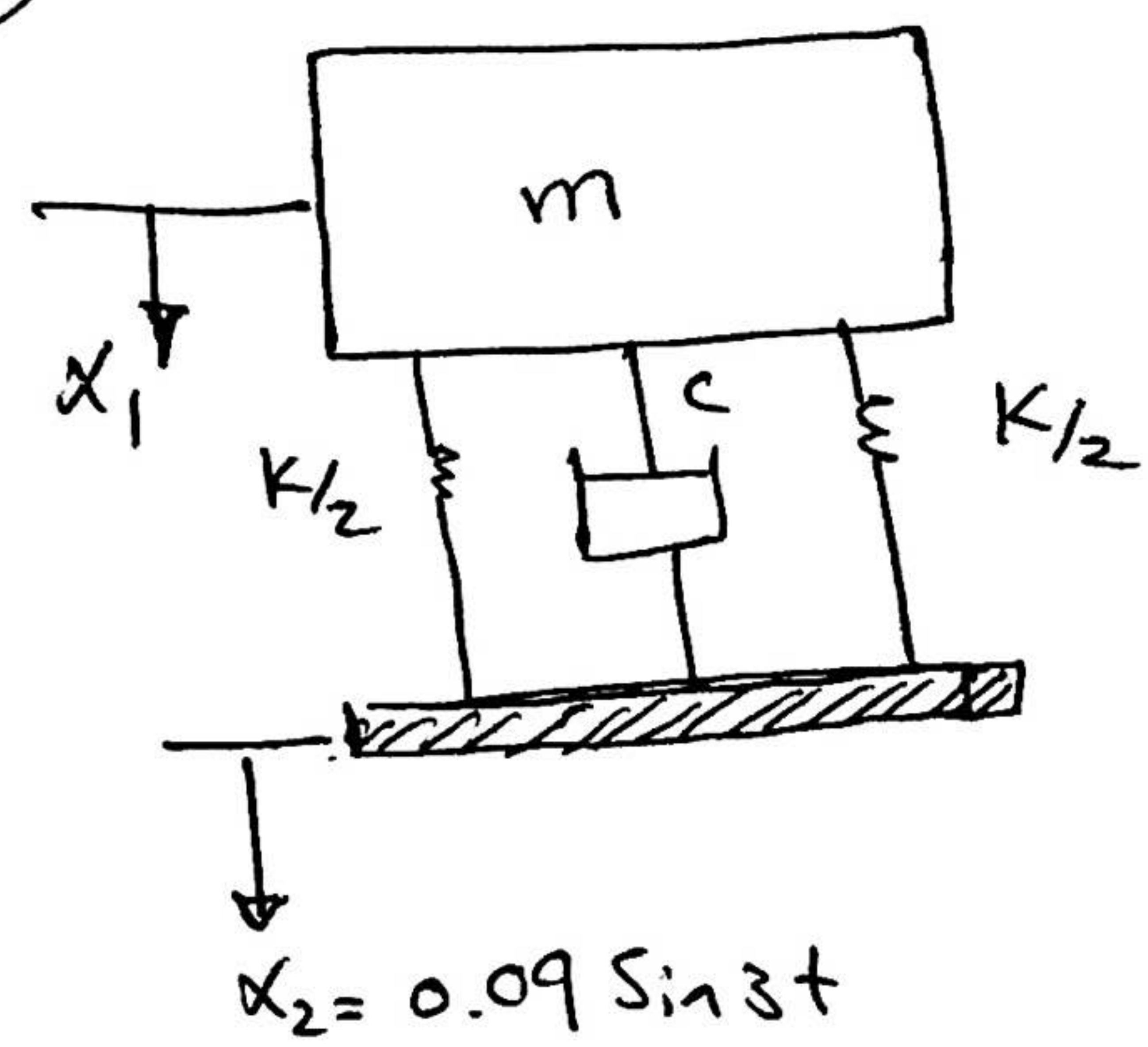
$X_2 = 0.09 \sin 3t$ ft

$K = 50$ Ib/in

$W = 3.86$ Ib

$\omega = 3$ rad/sec

$$\frac{Y}{X} = \sqrt{\frac{(1 + (2\xi \frac{\omega}{\omega_n})^2}{(1 - (\frac{\omega}{\omega_n})^2)^2 + (2\xi \frac{\omega}{\omega_n})^2}}$$



But $\xi = 0$

$$\therefore \frac{Y}{X} = \frac{1}{(1 - (\frac{\omega}{\omega_n})^2)}$$

$\therefore X_2 = 0.09 \sin 3t$ ft $\Leftrightarrow$ $Y = Y \sin \omega t$

$\therefore Y = 0.09$ ft

and $$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{50 * 12}{3.86/32.2}} = 70.74 \text{ rad/sec}$$

$$\therefore \frac{0.09}{X} = \frac{1}{(1 - (\frac{3}{70.74})^2)} \Rightarrow X = 0.09\,(0.9982)$$

$\therefore X = 0.0898$ ft

$Y = 0.09 \sin 3t \Rightarrow \dot{Y} = 0.09 * 3 . \cos 3t$

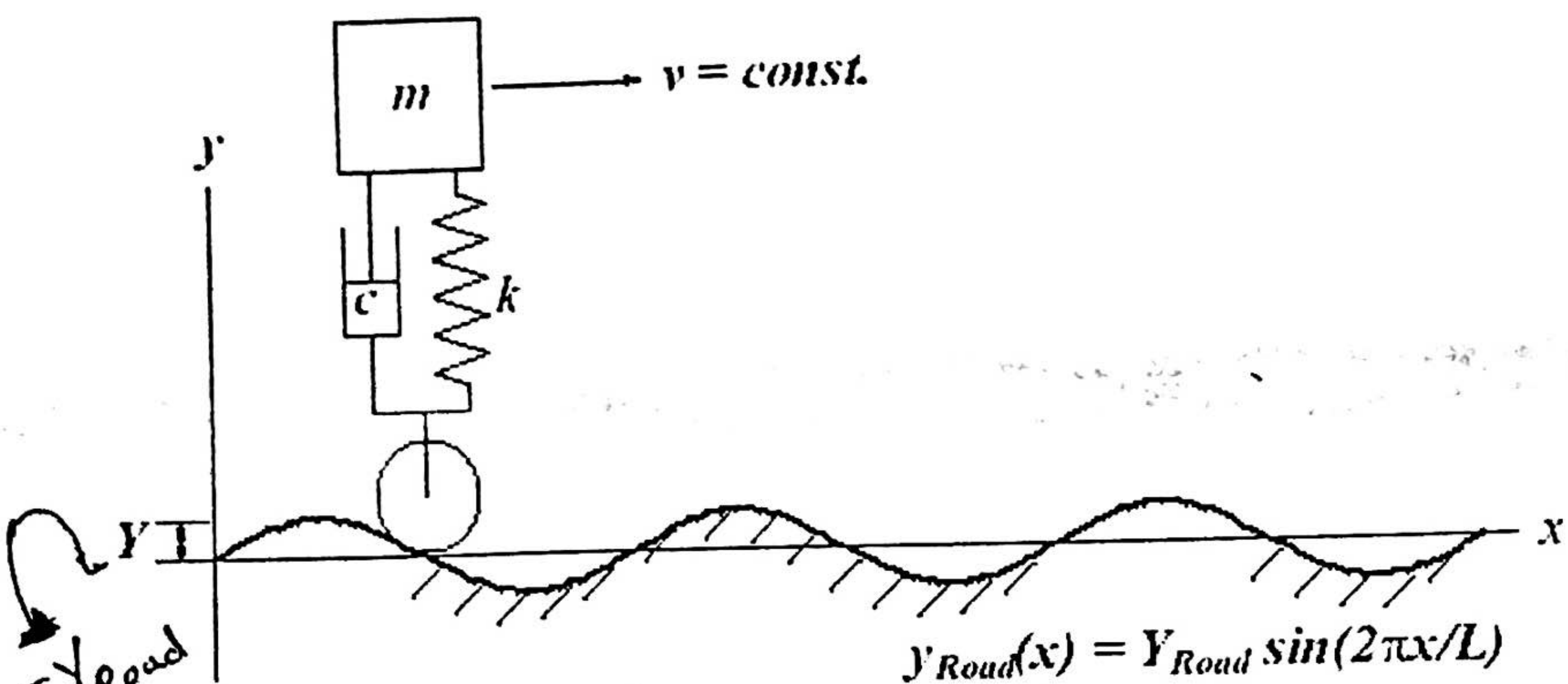
$\ddot{Y} = -0.09 * 3^2 . \sin 3t$ the Max acceleration occurs at $\sin 3t = 1$

$\therefore \ddot{Y} = -0.09 * 9 * 1 = -0.81$ rad/sec^2

طريق وعر فوق متنقلة عربة

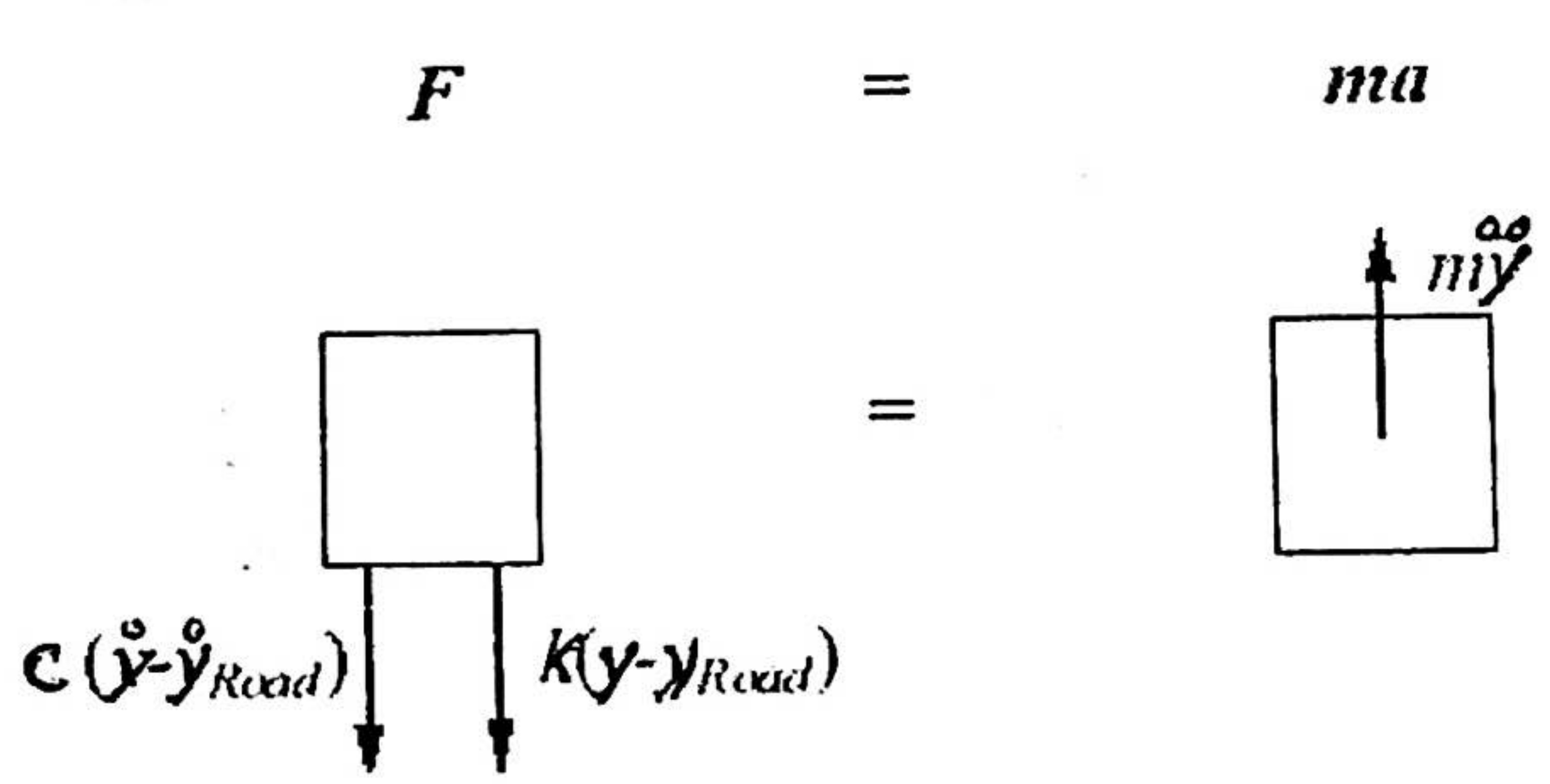
Vehicle Traveling over a Bumpy Road

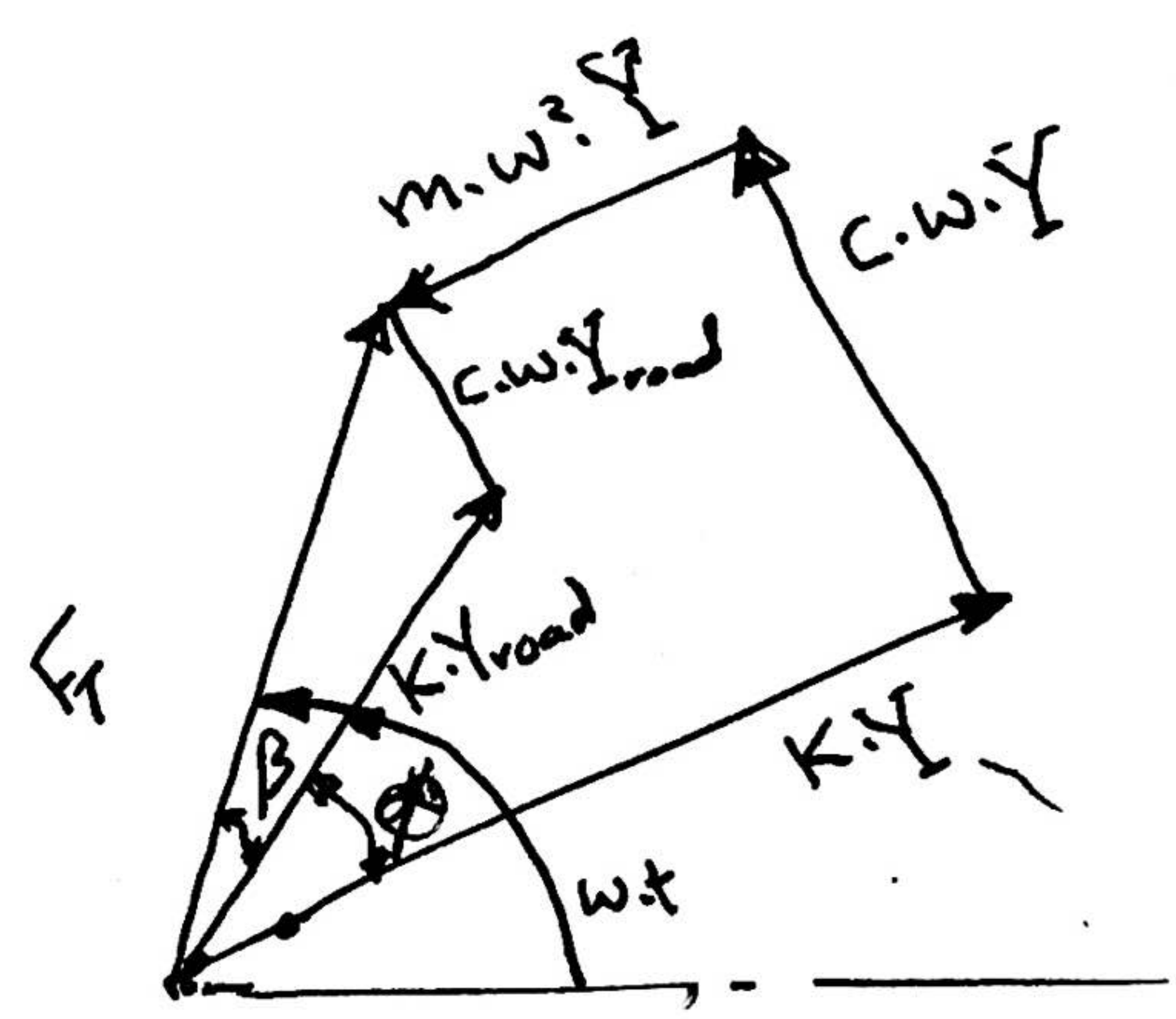
Consider a simple model of a vehicle moving over a bumpy road as illustrated in the following figure. Assume that the vehicle vibrates only in the vertical direction, the stiffness and damping effects of the tire can be neglected, and the tire has good traction and never leaves the road surface.



$y_{Road} = Y_{Road}\cdot\sin\omega t$ But $\omega = \frac{2\pi v}{L}$

The free body diagram of this moving-base system can be illustrated as,





The equation of motion is thus,

$$m\ddot{y}+c\,(\dot{y}-\dot{y}_{Road})+k(y-y_{Road})=0$$

Suppose that the vehicle is traveling at a constant speed, v, and the road roughness can be approximated by the equation,

$$y_{Road}(x) = Y_{Road}\sin\left(\frac{2\pi x}{L}\right)$$

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The road roughness can then be rewritten in terms of time (instead of position),

$$y_{Road}(t) = Y_{Road} \sin\left(\frac{2\pi v}{L}t\right) = Y_{Road} \sin \omega t$$

where $\omega = \dfrac{2\pi v}{L}$

The harmonic moving base system is then equivalent to a harmonic vibration excitation with the equation of motion,

$$m\ddot{y} + c\,\dot{y} + ky = c\,\omega Y_{Road} \cos \omega t + kY_{Road} \sin \omega t$$

$$\frac{Y}{Y_{Road}} = \sqrt{\frac{(1+(2\xi\frac{\omega}{\omega_n})^2}{(1-(\frac{\omega}{\omega_n})^2)^2+(2\xi\frac{\omega}{\omega_n})^2}}$$

; Y = amplitude of the system (m)

Y_{Road} = amplitude of the Road (m)

$$\cancel{\emptyset} = \tan^{-1} \frac{2\xi\frac{\omega}{\omega_n}}{(1-(\frac{\omega}{\omega_n})^2)}$$

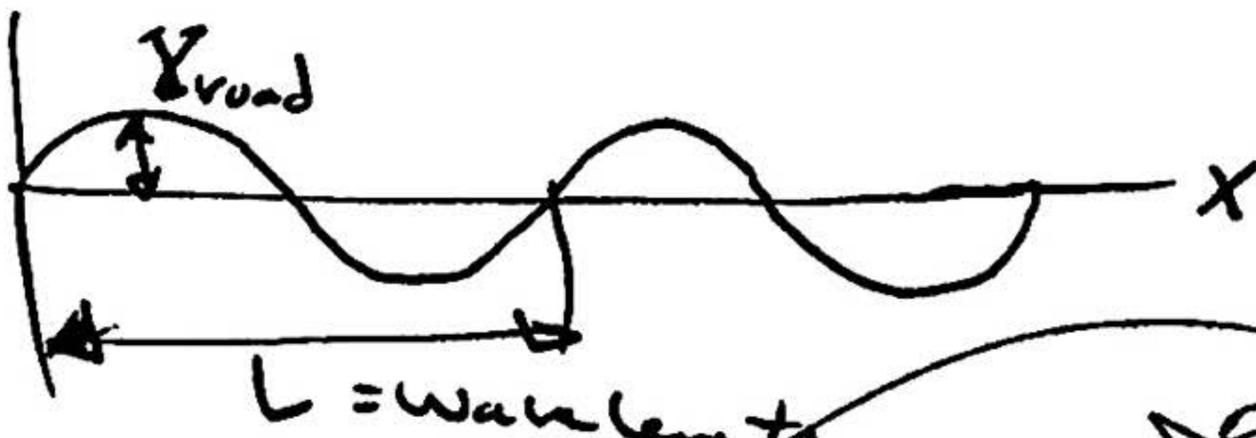
$$\beta = \beta = \tan^{-1} \frac{2\xi\,\omega/\omega_n}{1}$$

① $\omega = \dfrac{2\pi \cdot v}{L}$ v: The velocity of vechile m/sec
L: wave length.

② $y_{road} = Y_{road} \cdot \sin \omega t$ and $\omega = \dfrac{2\pi \cdot v}{L}$; Y_{road} = The amplitude o road (m)

③ Describe of motion of road by:-

$$y_{road} = Y_{road} \cdot \sin \frac{2\pi \cdot x}{L}$$



④ Max amplitude occurs at

$$\omega = \omega_n \sqrt{(1-2\xi^2}$$

⑤ $$\frac{Y}{Y_{road}} = \sqrt{\frac{(1+(2\xi\cdot\frac{\omega}{\omega_n})^2}{(1-(\frac{\omega}{\omega_n})^2+(2\xi\cdot\frac{\omega}{\omega_n})^2}}$$

Example: For the system shown in figure, ($m = 1200$ kg), ($k = 400$ kN/m), ($\xi = 0.5$), ($Y = 0.05$ m), and wave length (6 m). Find X? $V = 100$ Km/hr

solution:

$$\frac{X}{Y} = \frac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \sqrt{\frac{1 + (2\xi \cdot \frac{\omega}{\omega_n})^2}{(1 - (\frac{\omega}{\omega_n})^2)^2 + (2\xi \frac{\omega}{\omega_n})^2}}$$

$\xi = C/C_c \Rightarrow C = \xi \cdot C_c$

But $C_c = 2m\omega_n = 2m(k/m)^{1/2}$

$\therefore C = 0.5 * 2 * 1200 * (400*10^3 / 1200)$

$= 21909$ N.S/m.

$$\therefore X = Y * \frac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$\omega = 2\pi F = 2\pi * \left(\frac{100*1000}{3600}\right) * \frac{1}{6} = 29.0887$ rad/sec.

$\omega = 2\pi \cdot \frac{V}{L}$

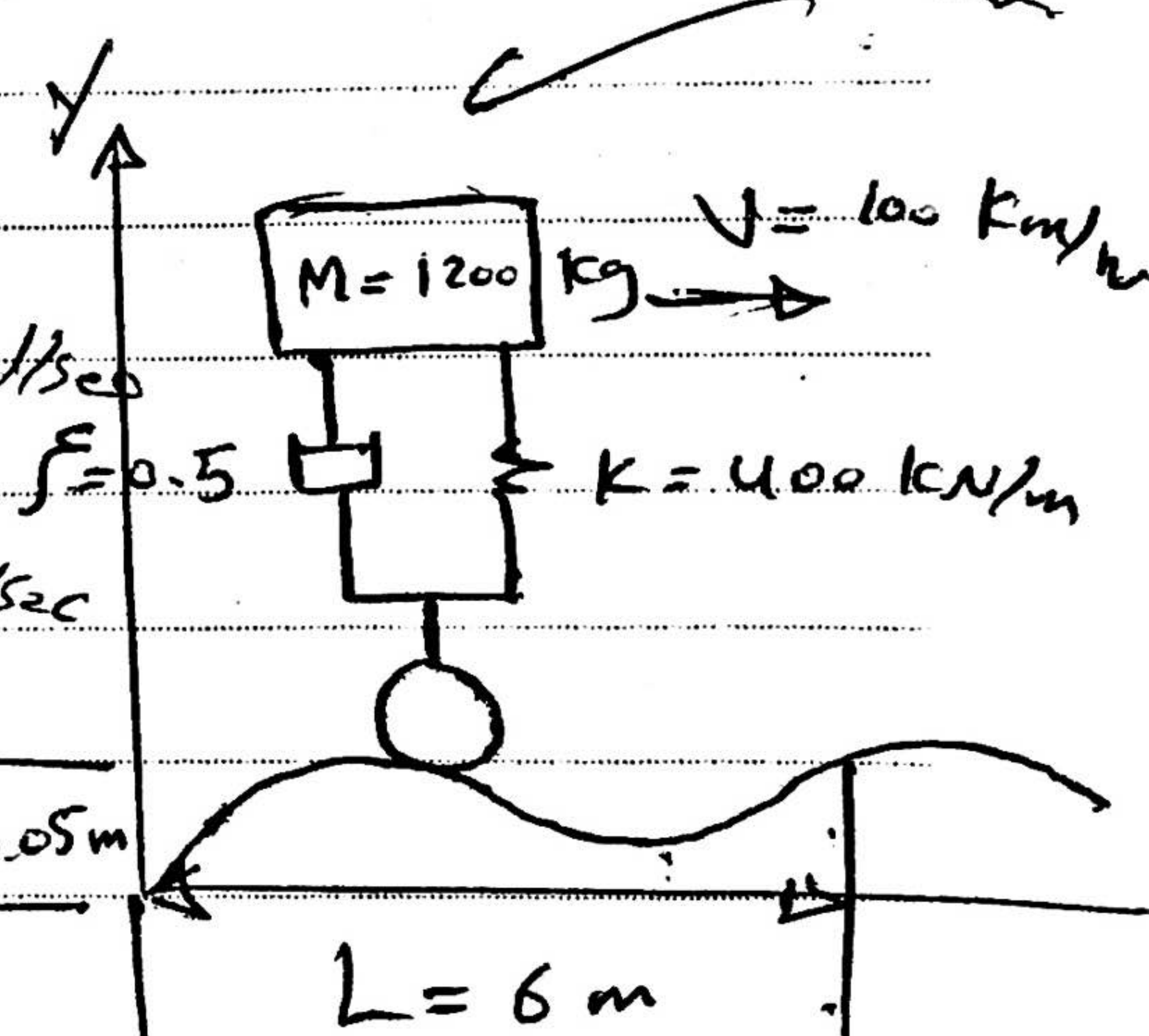
$$X = 0.05 * \frac{\sqrt{(400*10^3)^2 + (21909*29.0887)^2}}{\sqrt{(400*10^3 - 1200*29.0887^2)^2 + (21909*29.0887)^2}}$$

$= 0.042466$ m.

Sol:-

$\omega = \frac{2\pi \cdot V}{L} = \frac{2\pi * 100/3.6}{6} = 29$ rad/sec

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{400*10^3}{1200}} = 18.25$ rad/sec

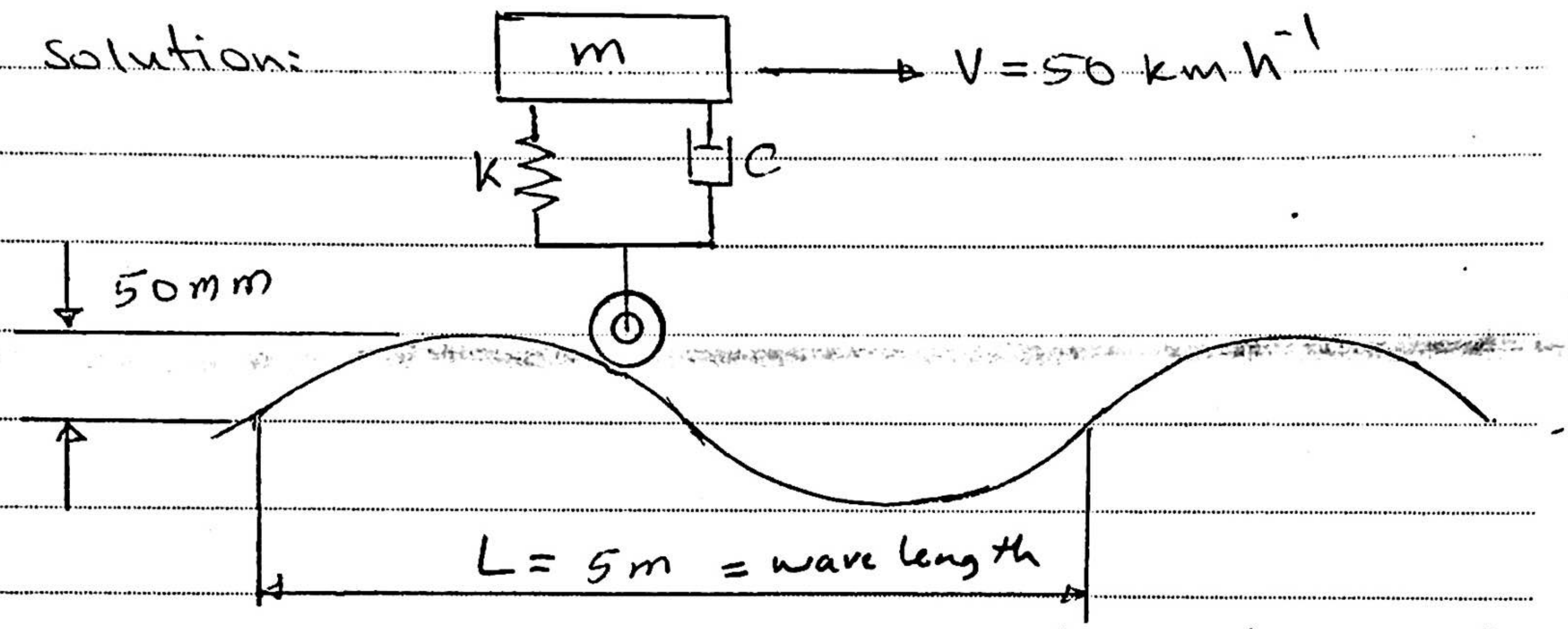


$$Y = Y_{Road} \cdot \sqrt{\frac{1 + (2\xi \cdot \frac{\omega}{\omega_n})^2}{(1 - (\frac{\omega}{\omega_n})^2)^2 + (2\xi \frac{\omega}{\omega_n})^2}}$$

$= 0.05 * \frac{1.83}{2.2} = 0.042$ m

Example: A trailer of mass 1000 kg is pulled with a constant speed of 50 km h^{-1} over a bumpy road which may be modelled as a sine wave of wave length 5 m and amplitude 50mm. Assume that the effective stiffness of the suspension is 350 kN m^{-1} and that the damping ratio 0.5. Determine the amplitude of the motion of the trailer and find the speed of the trailer at which this amplitude becomes a maximum.

Solution:



This is forced vibration due to ground excitation in effect. Imagine that the trailer is stationery and the ground underneath is being moved to the left at a speed of

$50\ km\,h^{-1} = 50 \times 1000/3600 = 13.9\ m\,s^{-1}$.

One cycle will have occurred when the ground has moved 5 m. This will take 5/13.9 s. So the number of cycles per second is $13.9/5 = 2.78$.

$m = 1000\ kg$, $K = 350\ kN\,m^{-1}$, $\xi = 0.5$

$\omega = 2\pi f = 2\pi \times 2.78 = 17.47\ rad\,s^{-1}$

$Y = 50\ mm = 0.05\ m$

$\omega = \dfrac{2\pi \cdot V}{L}$

$\therefore \omega = \dfrac{2\pi \times 50 \times \frac{1000}{3600}}{5}$

$\omega = 17.45\ rad/sec$

$\therefore \omega_n = (k/m)^{1/2} = ((350*1000)/1000)^{1/2} = 18.71 \ rad\,s^{-1}$

$\therefore r = \omega/\omega_n = 17.47/18.71 = 0.934$

$$\therefore F = Y_{road}\left[\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}\right]^{1/2} = 0.05\left[\frac{1+(2*0.5*0.934)^2}{(1-0.934^2)^2+(2*0.5*0.934)^2}\right]^{1/2}$$

$$= 0.05\left[\frac{1+0.872}{0.016+0.872}\right]^{1/2} = 0.05*\sqrt{2.108} = 0.073\ m$$

$$= 73\ mm$$

The maximum amplitude occurs when

$\omega = \omega_n\sqrt{1-2\xi^2} = 18.71*\sqrt{1-2*0.5^2} = 13.23 \ rad\,s^{-1}$

The driving frequency (in HZ) is the speed, V, divided by the wavelength, so

$$\omega = \frac{2\pi * V}{5} \Rightarrow V = \frac{5*\omega}{2\pi} = \frac{5*13.23}{2\pi} = 10.53\ m\,s^{-1}$$

The speed at which the amplitude of the trailer motion reaches a maximum is

$$\frac{10.53*3600}{1000} = 38\ km\,h^{-1}.$$

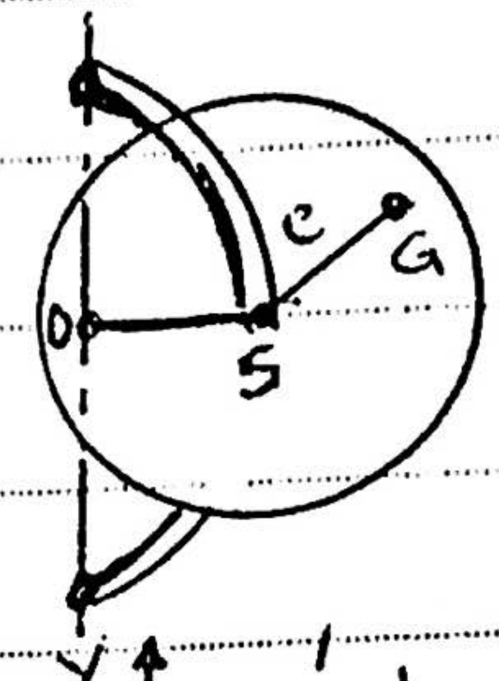
$$\omega = \frac{2\pi . V}{\lambda}$$

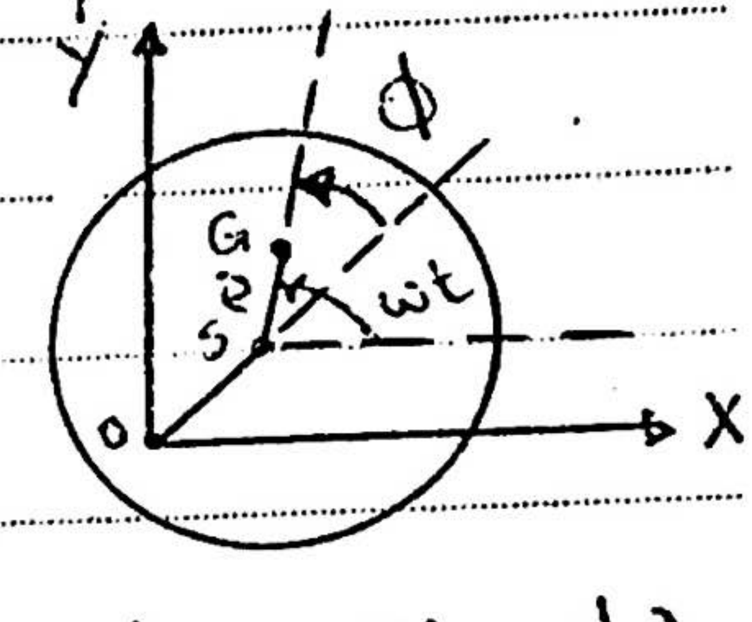
الأعمدة الدوارة   التدويم الغرسي

⊙ Whirling of Rotating Shafts.

Whirling is defined as the rotating of the plane made by the bent shaft and the line of centers of the bearing.

Consider here a single disk of mass (m) symmetrically located on a shaft supported by two bearing as shown in the figure. The center of mass (G) of the disk is at a distance (e)(eccentricity) from the geometric center (S) of the disk. The center line of the bearings intersects the plane of the disk at (O), and the shaft center is deflected by $r = OS$.





∴ the coordinates of (S) are (x_s, y_s).

the coordinates of (G) are $(x_s + e\cos\omega t,\ y_s + e\sin\omega t)$.

∴ the equations of motion in the x and y direction are;

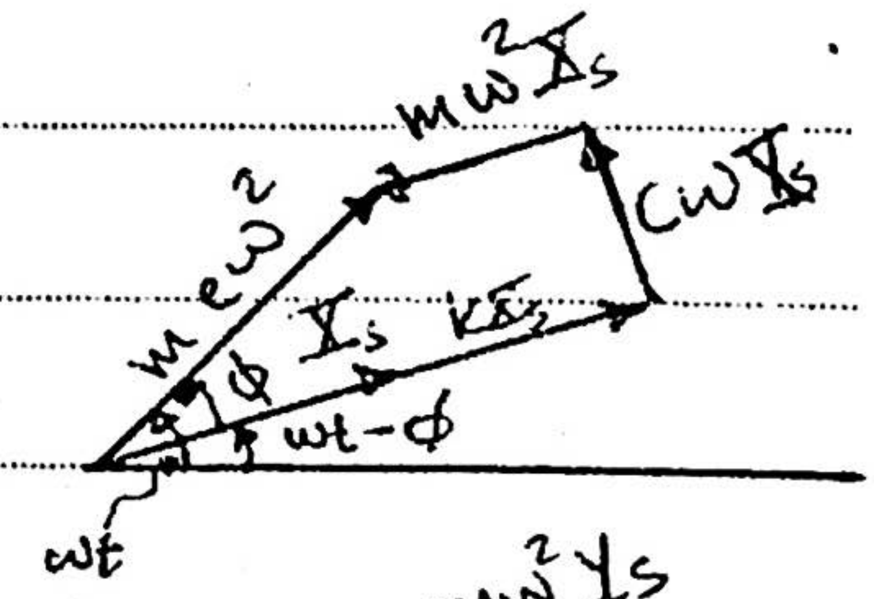
$$m\frac{d^2}{dt^2}(x_s + e\cos\omega t) = -k x_s - c\dot{x}_s$$

and $$m\frac{d^2}{dt^2}(y_s + e\sin\omega t) = -k y_s - c\dot{y}_s$$

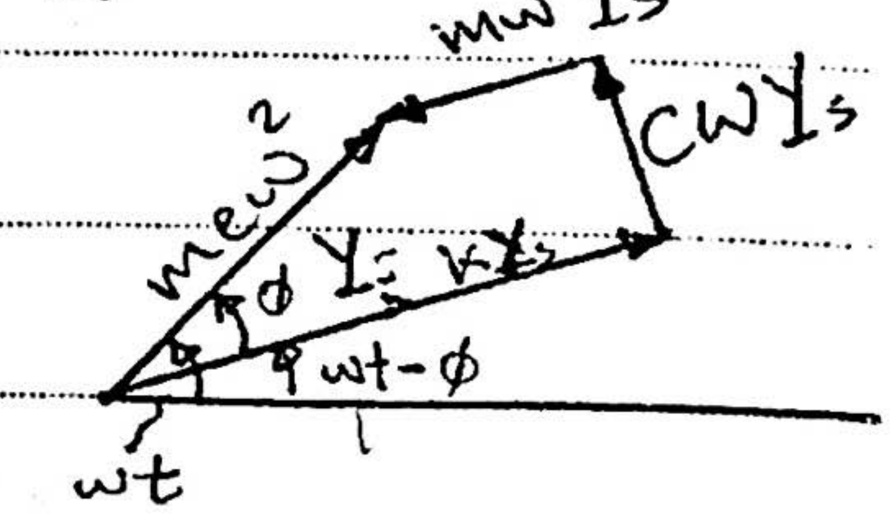
OR $$m\ddot{x}_s + c\dot{x}_s + kx_s = me\omega^2 . \cos\omega t.$$

$$m\ddot{y}_s + c\dot{y}_s + ky_s = me\omega^2 . \sin\omega t.$$

$$\Rightarrow\quad x_s = \frac{me\omega^2}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} . \cos(\omega t - \phi)$$



$$y_s = \frac{me\omega^2}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} . \sin(\omega t - \phi)$$



$$x_s = X_s\cos(\omega t - \phi)\ ,\ y_s = Y_s\sin(\omega t - \phi).$$

لتحقيق التوازن الديناميكي للكتل الدوارة يجب أن يتطابق المحور الهندسي للحركة أو الجسم مع مركز ثقله. لأن الأختلافات المركزية يسبب عدم الاتزان والاهتزاز

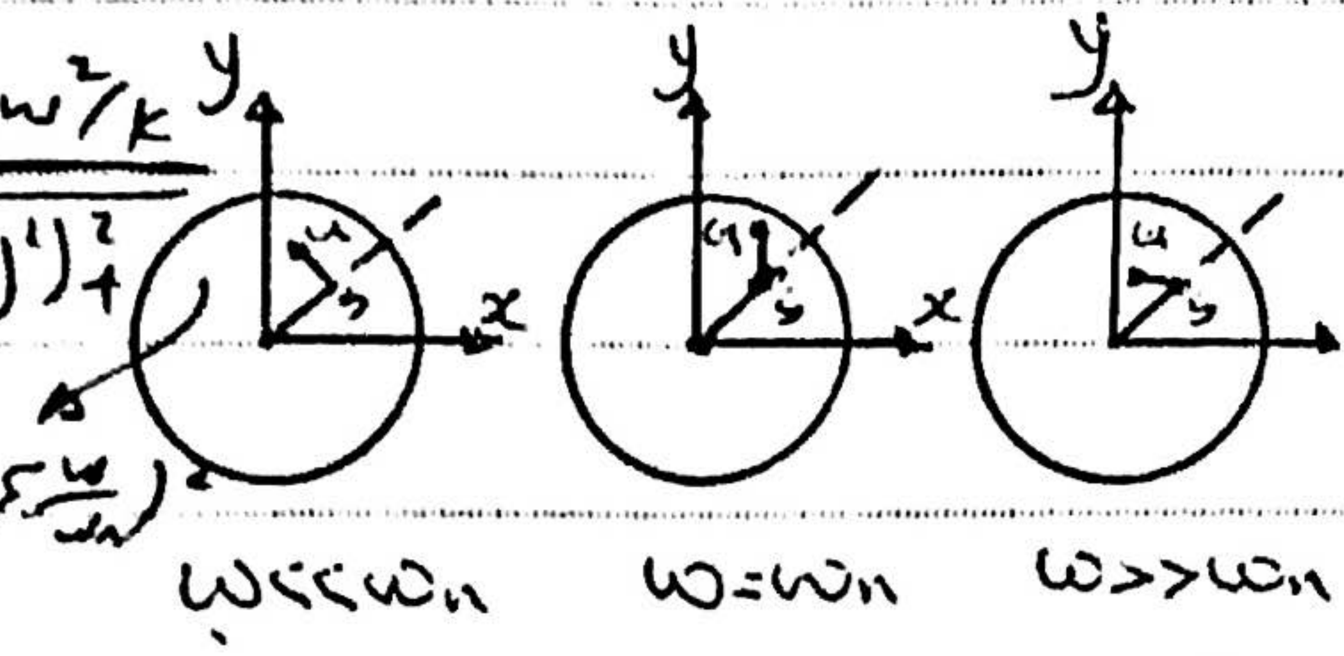
and $OS = r = \sqrt{x_s^2 + y_s^2}$

$$= \sqrt{\left(\frac{me\omega^2}{\sqrt{(K-m\omega^2)^2 + (c\omega)^2}}\right)^2 \cdot \left(\cos^2(\omega t-\phi) + \sin^2(\omega t-\phi)\right)}$$

Note/ $\cos^2(\omega t-\phi) + \sin^2(\omega t-\phi) = 1$

$$\therefore\ r = \frac{me\omega^2}{\sqrt{(K-m\omega^2)^2 + (c\omega)^2}} = \frac{me.\omega^2/K}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$\tan\phi = \frac{c\omega}{K-m\omega^2} = \frac{2\xi.\omega/\omega_n}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}$$



Example: A solid disk of weight (10 Ib) is keyed to مقفول
the center of a (0.5 in) steel shaft (2 ft) between bearing.
Determine the lowest critical speed. (Assume shaft to be
simply supported at the bearing). and $\xi = 0$ $E_s = 30 * 10^6$ Ib/in², Note 1 ft = 12 in

Solutions

$R = 0.5$ in $\Rightarrow D = 2(0.5) = 1$ in

$$K = \frac{48EI}{L^3}$$

But $I = \frac{\pi D^4}{64}$

$$\Rightarrow K = \frac{3\pi E D^4}{4L^3} = \frac{3 * \pi * 30 * 10^6 * (1)^4}{4 * (2*12)^3} = 5113.2\ \text{Ib/in}$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{3\pi E D^4}{4L^3 . m}} = 22.6\ \text{rad/sec}$$

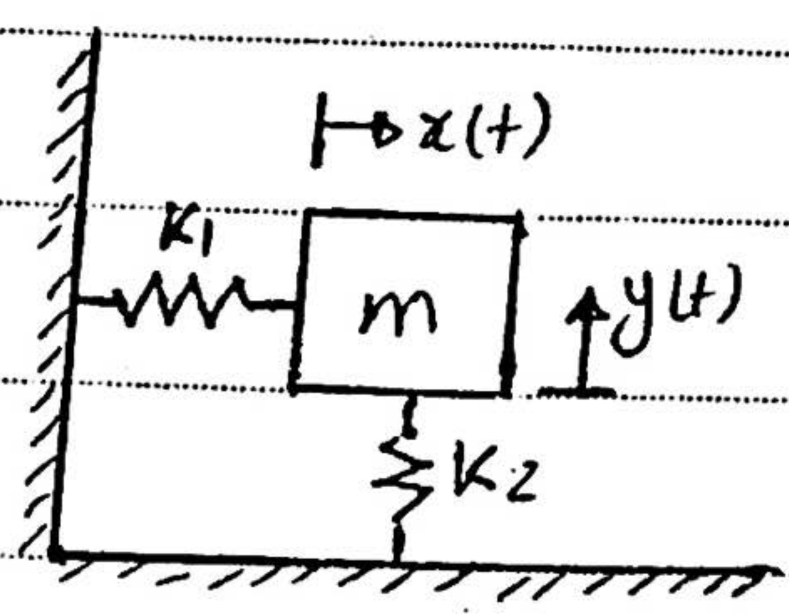
Critical Speed.

H.W// by using the equation of forced vibration of damping system of whirling of rotating shaft Determine the equation of displacement and the critical frequency. Assume shaft to be Simply Supported beam

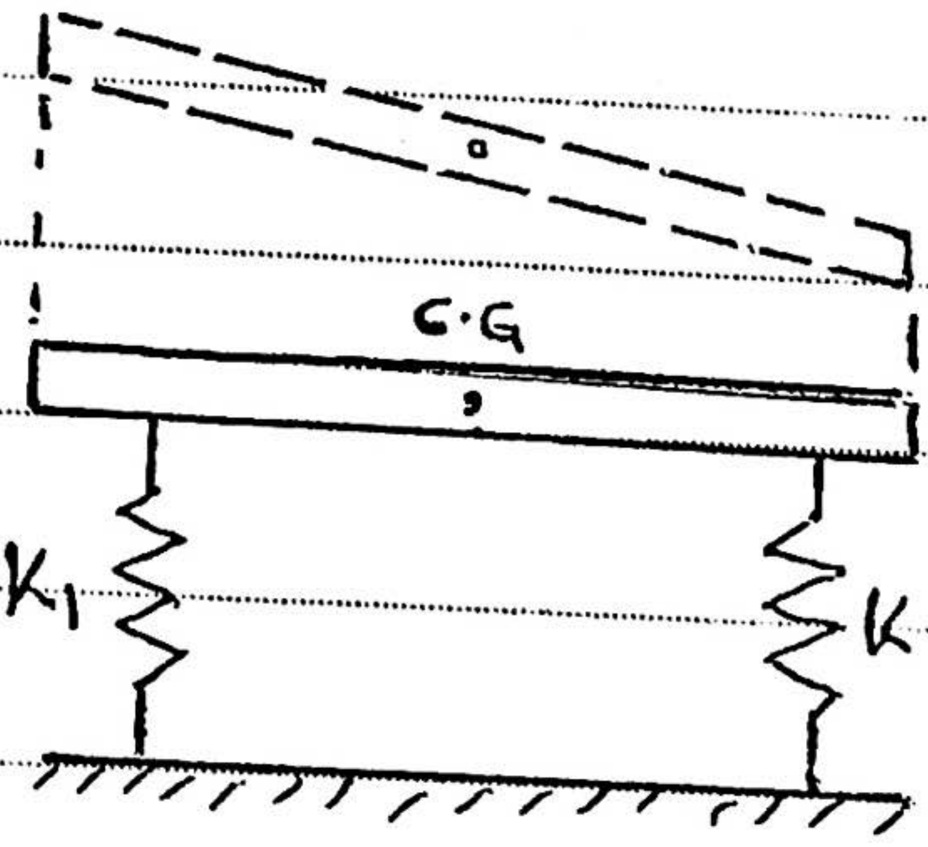
Chapter 4
Systems with Two DOF

Introduction:

- An N-DOF system will have N natural frequencies.
- Each natural frequency has a natural state of vibration Called its normal mode
- Eigenvalues and eigenvectors are related to the normal modes.
- Normal mode vibrations are undamped free vibrations that depend only on the mass and stiffness distribution in the system.
- Damping limits amplitude and forces the free vibrations to decay.
- Number of DOF of the system equal to the Number of masses in the system multiply by Number of position types of motion of each mass.



x, y



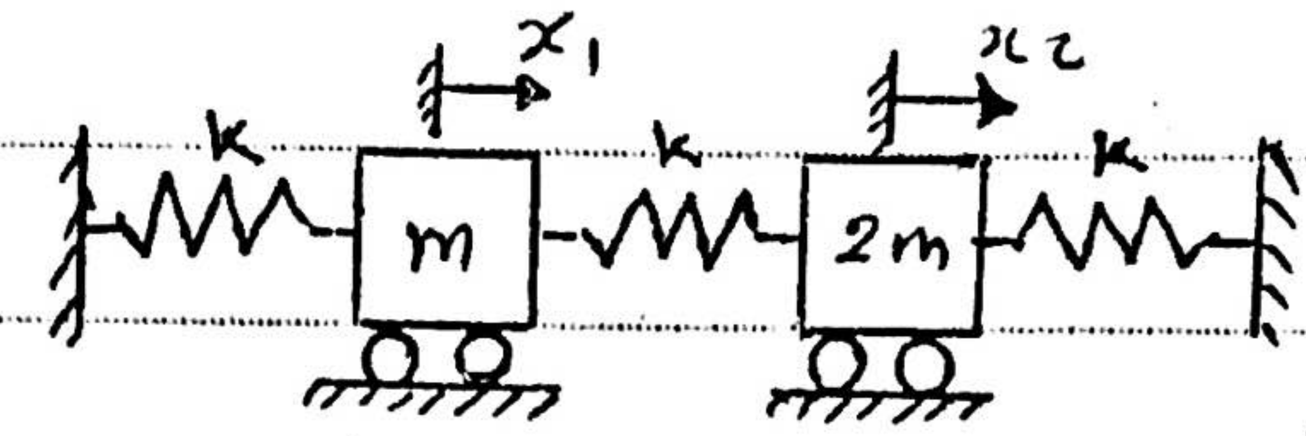
y, θ

تحليل النسق ...

Normal Mode Analysis:

If $x_1 > x_2$

$\overrightarrow{m\ddot{x}} = \overrightarrow{\sum F_x}$

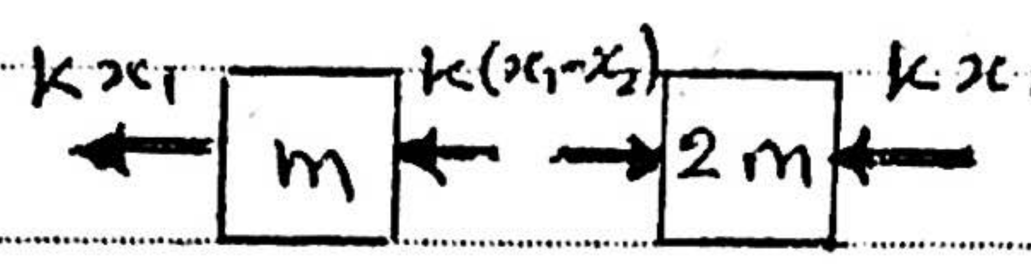


For (m) $\Rightarrow m\ddot{x}_1 + kx_1 + k(x_1 - x_2) = 0$

For (2m) $\Rightarrow 2m\ddot{x}_2 - k(x_1 - x_2) + kx_2 = 0$

let $x_1 = A_1 \sin\omega t$ or $A_1 e^{i\omega t}$

$x_2 = A_2 \sin\omega t$ or $A_2 e^{i\omega t}$



$\Rightarrow \dot{x}_1 = iA_1\omega e^{i\omega t}$; $\dot{x}_2 = iA_2 e^{i\omega t}$

$\ddot{x}_1 = -A_1\omega^2 e^{i\omega t}$ $\quad \ddot{x}_2 = -A_2\omega^2 e^{i\omega t}$

$\therefore -mA_1\omega^2 e^{i\omega t} + kA_1 e^{i\omega t} + kA_1 e^{i\omega t} - kA_2 e^{i\omega t} = 0$

$-2mA_2\omega^2 e^{i\omega t} - kA_1 e^{i\omega t} + kA_2 e^{i\omega t} + kA_2 e^{i\omega t} = 0$

$\Rightarrow (-mA_1\omega^2 + 2kA_1 - kA_2)e^{i\omega t} = 0$

$(-2mA_2\omega^2 - kA_1 + 2kA_2)e^{i\omega t} = 0$

$\Rightarrow (-m\omega^2 + 2k)A_1 - kA_2 = 0 \quad \ldots (1)$

$-kA_1 + (-2m\omega^2 + 2k)A_2 = 0 \quad \ldots (2)$

or $\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - 2m\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$

$\Rightarrow \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - 2m\omega^2 \end{vmatrix} = 0$

If we let $\omega^2 = \lambda$

$\Rightarrow \begin{vmatrix} 2k - \lambda m & -k \\ -k & 2k - 2\lambda m \end{vmatrix} = 0$

or $(2k - \lambda m)(2k - 2\lambda m) - k^2 = 0$

$4k^2 - 4km\lambda - 2km\lambda + 2\lambda^2 m^2 - k^2 = 0$

$2m^2\lambda^2 - 6km\lambda + 3k^2 = 0$

$$\Rightarrow \lambda^2 - \left(3\frac{k}{m}\right)\lambda + \frac{3}{2}\left(\frac{k}{m}\right)^2 = 0$$

$$\lambda_{1,2} = \frac{3\frac{k}{m} \pm \sqrt{9\left(\frac{k}{m}\right)^2 - 6\left(\frac{k}{m}\right)^2}}{2} = \left(\frac{3}{2} \pm \frac{\sqrt{3}}{2}\right)\frac{k}{m}$$

$\therefore$ $\lambda_1 = 0.634\,\frac{k}{m}$
$\lambda_2 = 2.366\,\frac{k}{m}$ } القيمة المميزة eigenvalue of the system.

$\omega_1 = \sqrt{\lambda_1} = \sqrt{0.634\,k/m}$
$\omega_2 = \sqrt{\lambda_2} = \sqrt{2.366\,k/m}$ } natural frequencies of the system.

From equations (1) and (2) we can form the following relationship:

$$\left(\frac{A_1}{A_2}\right) = \frac{k}{2k - m\omega^2} = \frac{2k - 2m\omega^2}{k}$$

For ω_1: $\left(\frac{A_1}{A_2}\right)^{(1)} = \frac{k}{2k - m(0.634)\frac{k}{m}} = \frac{1}{2 - 0.634} = 0.732$

For ω_2: $\left(\frac{A_1}{A_2}\right)^{(2)} = \frac{k}{2k - m(2.366)\frac{k}{m}} = \frac{1}{2 - 2.366} = -2.732$

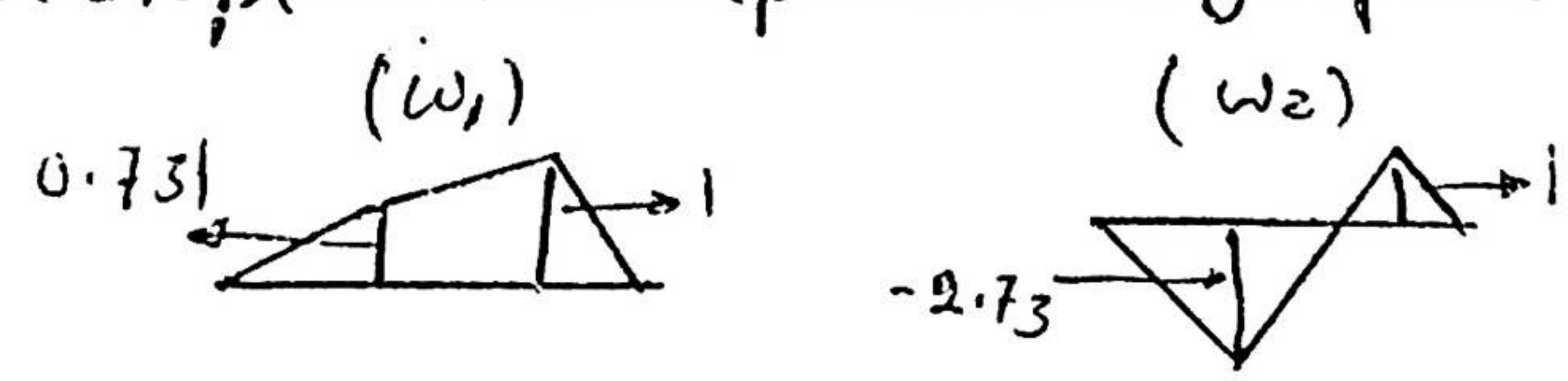
If one of the amplitudes is chosen $= 1$, then the amplitude ratio are normalized. These are called the (normal modes), $\Phi_i(x)$, or (mode shapes) for the system:

$$\Phi_1(x) = \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \; ; \; \Phi_2(x) = \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix}$$

The normal mode are also the (eigen vectors) for the system and the normal mode oscillations are given by:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(1)} = A_1 \begin{Bmatrix} 0.732 \\ 1.00 \end{Bmatrix} \sin(\omega_1 t + \phi_1) \; ; \; \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(2)} = A_2 \begin{Bmatrix} -2.73 \\ 1.00 \end{Bmatrix} \sin(\omega_2 t + \phi_2)$$

The mode shapes can be represented graphically as follows:

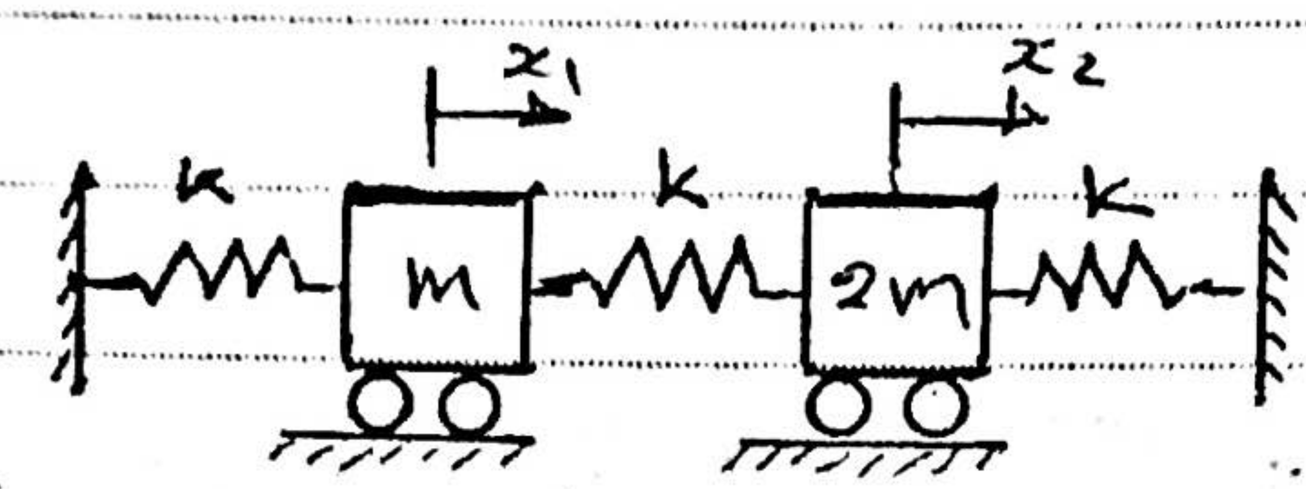


Initial Conditions:

Once we know the modal frequencies and mode shapes, we can determine the free vibration of the system for any initial conditions as a sum of the normal modes:

∴ From solution

$$\omega_1 = \sqrt{0.634\frac{k}{m}} \quad ; \quad \Phi_1 = \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix}$$



$$\omega_2 = \sqrt{2.366\frac{k}{m}} \quad ; \quad \Phi_2 = \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix}$$

For free vibrations in either of the normal modes also eigenvectors

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(i)} = C_i \Phi_i \sin(\omega_i t + \phi_{oi}) \quad ; \quad i = 1, 2$$

Where: C_i and ϕ_{oi} Come from initial Conditions

Φ_i assures that the Correct amplitude ratio is maintained.

For initial conditions in general, both modes are present:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = C_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \sin(\omega_1 t + \phi_{o1}) + C_2 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \phi_{o2})$$

Note that there are four Constant and two equations. The four Constants are:

$C_1, C_2 \Rightarrow$ define the Contribution of each mode, and

$\phi_1, \phi_2 \Rightarrow$ describe the phase relationships.

We Can differentiate to get two more equations:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \omega_1 C_1 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \cos(\omega_1 t + \phi_1) + \omega_2 C_2 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \cos(\omega_2 t + \phi_2)$$

To solve the four equations, we can take at $t = 0$, $x_1(0) = 2$; $x_2(0) = 4$; $\dot{x}_1(0) = 0$; $\dot{x}_2(0) = 0$.

$$\Rightarrow \quad 2 = 0.732\, C_1 \sin\phi_1 - 2.732\, C_2 \sin\phi_2 \quad \text{---}(1)$$

$$4 = C_1 \sin\phi_1 + C_2 \sin\phi_2 \quad]*(2.732)$$

$$10.928 = 2.732\, C_1 \sin\phi_1 + 2.732\, C_2 \sin\phi_2 \quad \text{---}(2)$$

add (1) and (2)

$$\boxed{12.928 = 3.464\, C_1 \sin\phi_1} \quad \text{----}(*)$$

and Sub. the initial Condition in velocity equations.

$$\Rightarrow \quad 0 = 0.732\, \omega_1 C_1 \cos\phi_1 - 2.732\, \omega_2 C_2 \cos\phi_2 \quad \text{---}(3)$$

$$0 = \omega_1 C_1 \cos\phi_1 + \omega_2 C_2 \cos\phi_2 \quad]*(2.732)$$

$$0 = 2.732\, \omega_1 C_1 \cos\phi_1 + 2.732\, \omega_2 C_2 \cos\phi_1 \quad \text{---}(4)$$

add (3) and (4)

$$0 = 3.464\, \omega_1 C_1 \cos\phi_1$$

$$\Rightarrow \quad \cos\phi_1 = 0 \Rightarrow \phi_1 = \cos^{-1} 0 = 90 \quad \text{ANS}$$

Sub. in (3) or (4) $\Rightarrow \phi_2 = 90$ ANS

Sub. in (*) $\Rightarrow 12.928 = 3.464\, C_1$

$$\therefore \Rightarrow C_1 = 3.732$$

Sub. in (1) or (2) $\Rightarrow C_2 = 0.26787$

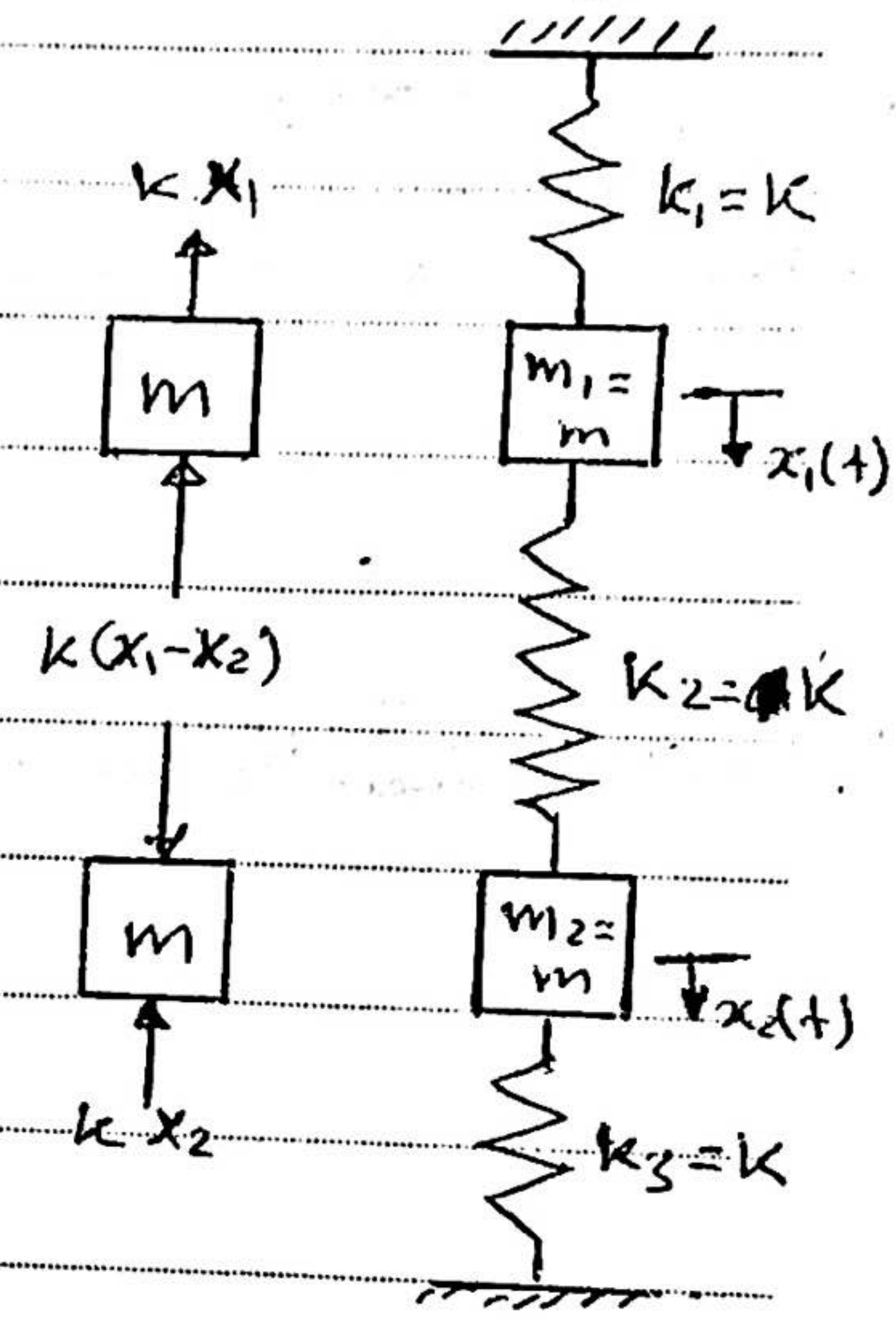
$$\Rightarrow \therefore \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 3.732 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \sin(\omega_1 t + 90) + 0.26787 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \sin(\omega_2 t + 90)$$

$$\Rightarrow \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 3.732 \begin{Bmatrix} 0.732 \\ 1 \end{Bmatrix} \cos\sqrt{0.634 \frac{k}{m}}\, t + 0.26787 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \cos\sqrt{2.366 \frac{k}{m}}\, t .$$

Example: Find the natural frequencies and mode shapes of a spring-mass system, shown in Figure below, which is constrained to move in the vertical direction only. And determine the free vibration of the system under the following initial condition,
$x_1(0)=5$; $x_2(0)=0$; $\dot{x}_1(0)=0$; $\dot{x}_2(0)=0$. then take $n=1$.

solution:

$m\ddot{x} = \Sigma F \downarrow$



For (m_1): $m\ddot{x}_1 = -k(x_1-x_2) - kx_1$

$\Rightarrow \quad m\ddot{x}_1 + 2kx_1 - kx_2 = 0$

For (m_2): $m\ddot{x}_2 = k(x_1-x_2) - kx_2$

$\Rightarrow \quad m\ddot{x}_2 + 2kx_2 - kx_1 = 0$

Let $x_1 = Ae^{i\omega t}$, $x_2 = Be^{i\omega t}$

$$\begin{bmatrix} 2k-m\omega^2 & -k \\ -k & 2k-m\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 2k-m\omega^2 & -k \\ -k & 2k-m\omega^2 \end{vmatrix} = 0$$

$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$

$\Rightarrow \omega_1^2 = \frac{k}{m} \Rightarrow \frac{A}{B}\Big|_1 = 1$ and $\omega_2^2 = \frac{3k}{m} \Rightarrow \frac{A}{B}\Big|_2 = -1$

(*) under initial Conditions

$x_1(t) = A\sin(\omega_1 t + \phi_1) - B\sin(\omega_2 t + \phi_2)$

$x_2(t) = A\sin(\omega_1 t + \phi_1) + B\sin(\omega_2 t + \phi_2)$

I.C.s : $x_1(0)=5 \Rightarrow 5 = A\sin\phi_1 - B\sin\phi_2$ ---(1)

$x_2(0)=0 \Rightarrow 0 = A\sin\phi_1 + B\sin\phi_2$ ---(2)

$\dot{x}_1(0)=0 \Rightarrow 0 = \omega_1 A\cos\phi_1 - B\omega_2\cos\phi_2$ ---(3)

$\dot{x}_2(0)=0 \Rightarrow 0 = \omega_1 A\cos\phi_1 + B\omega_2\cos\phi_2$ ---(4)

ملاحظة مهمة جداً / في حالة تساوي النسبة لسوي (Normal Modes) ($\frac{A_1}{A_2} = \pm 1$)
فإن معادلة الحركة تصبح كما في أعلاه (*) عند التعويض في المعادلات أعلاه

After solution we find,

$\phi_1 = \phi_2 = 90$

and $A = 2.5$

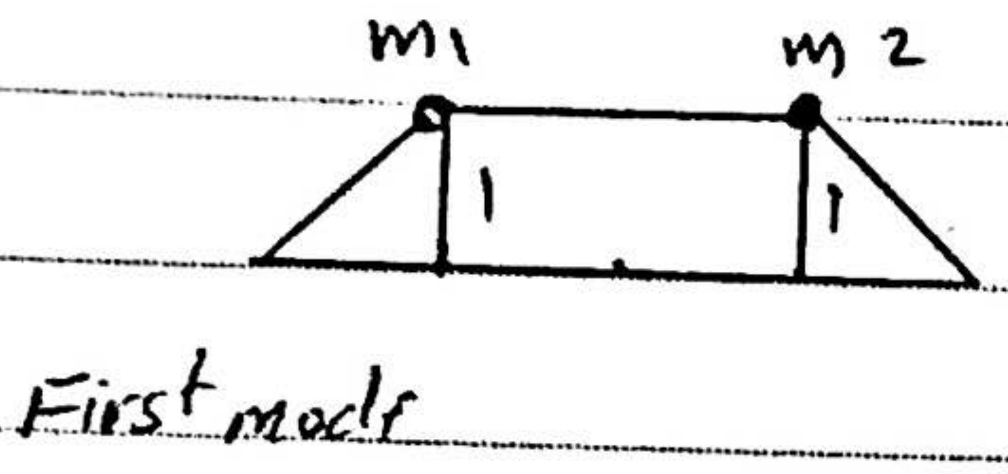
$B = -2.5$

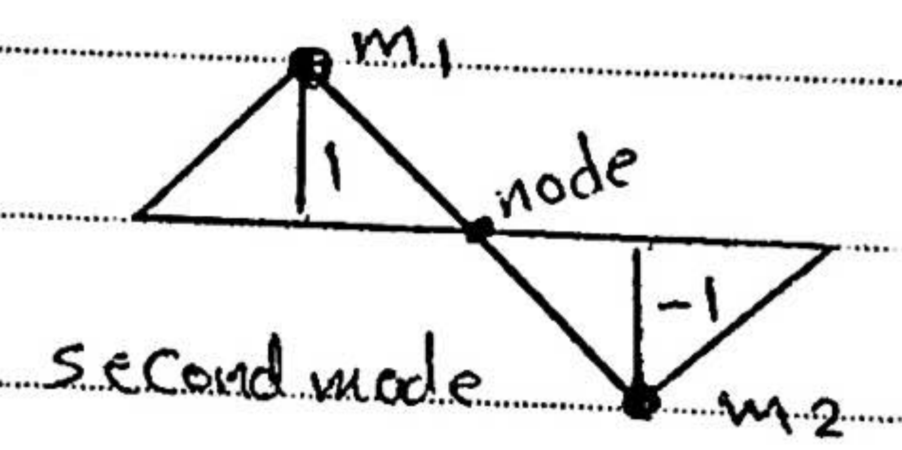
$\therefore \quad x_1(t) = 2.5 \sin\left(\sqrt{\frac{k}{m}}\, t + 90\right) + 2.5 \sin\left(\sqrt{\frac{3k}{m}}\, t + 90\right)$

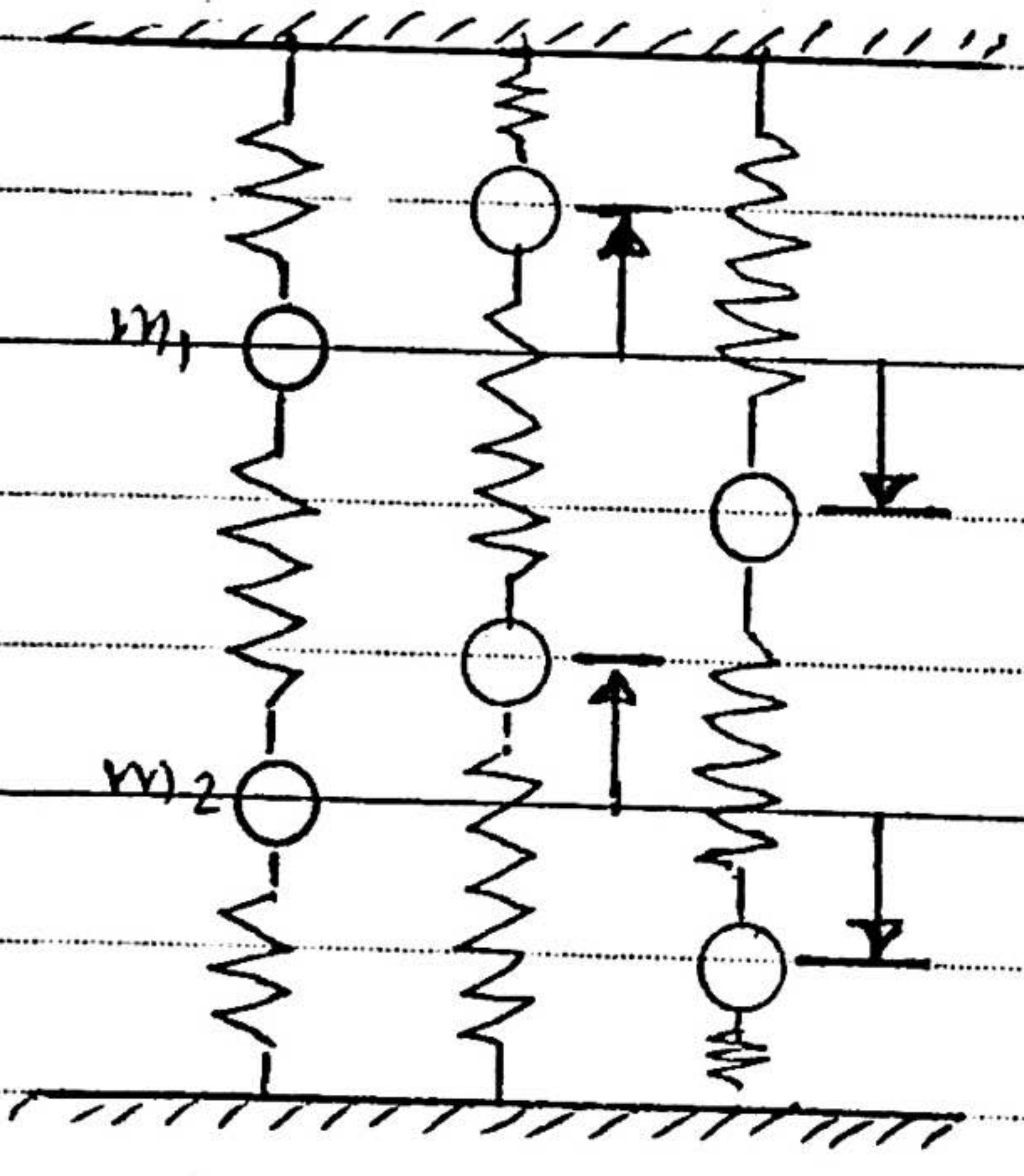
$x_2(t) = 2.5 \sin\left(\sqrt{\frac{k}{m}}\, t + 90\right) - 2.5 \sin\left(\sqrt{\frac{3k}{m}}\, t + 90\right)$

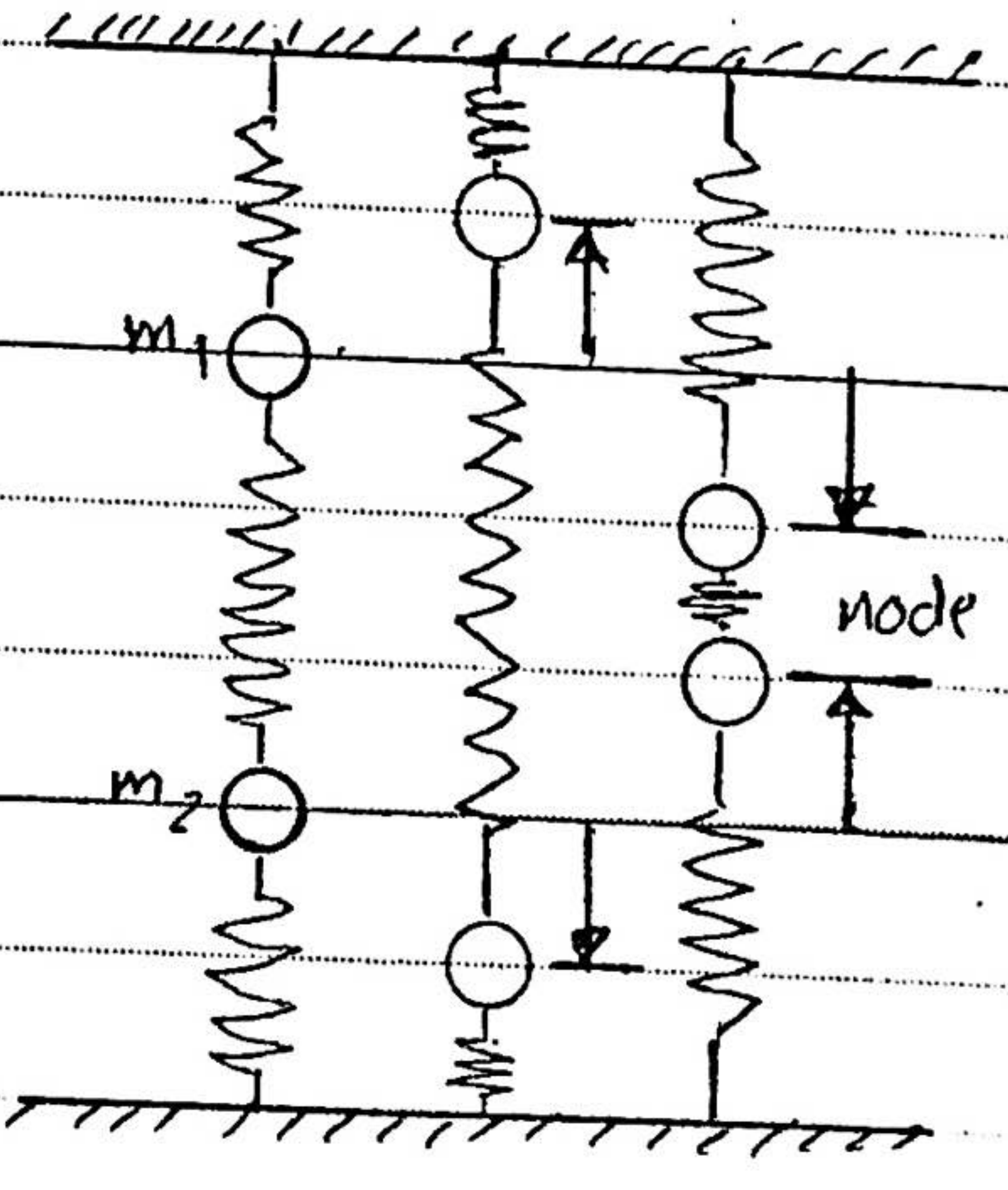
$\Rightarrow$ or,

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = 2.5 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos\sqrt{\frac{k}{m}}\, t - 2.5 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \cos\sqrt{\frac{3k}{m}}\, t .$$





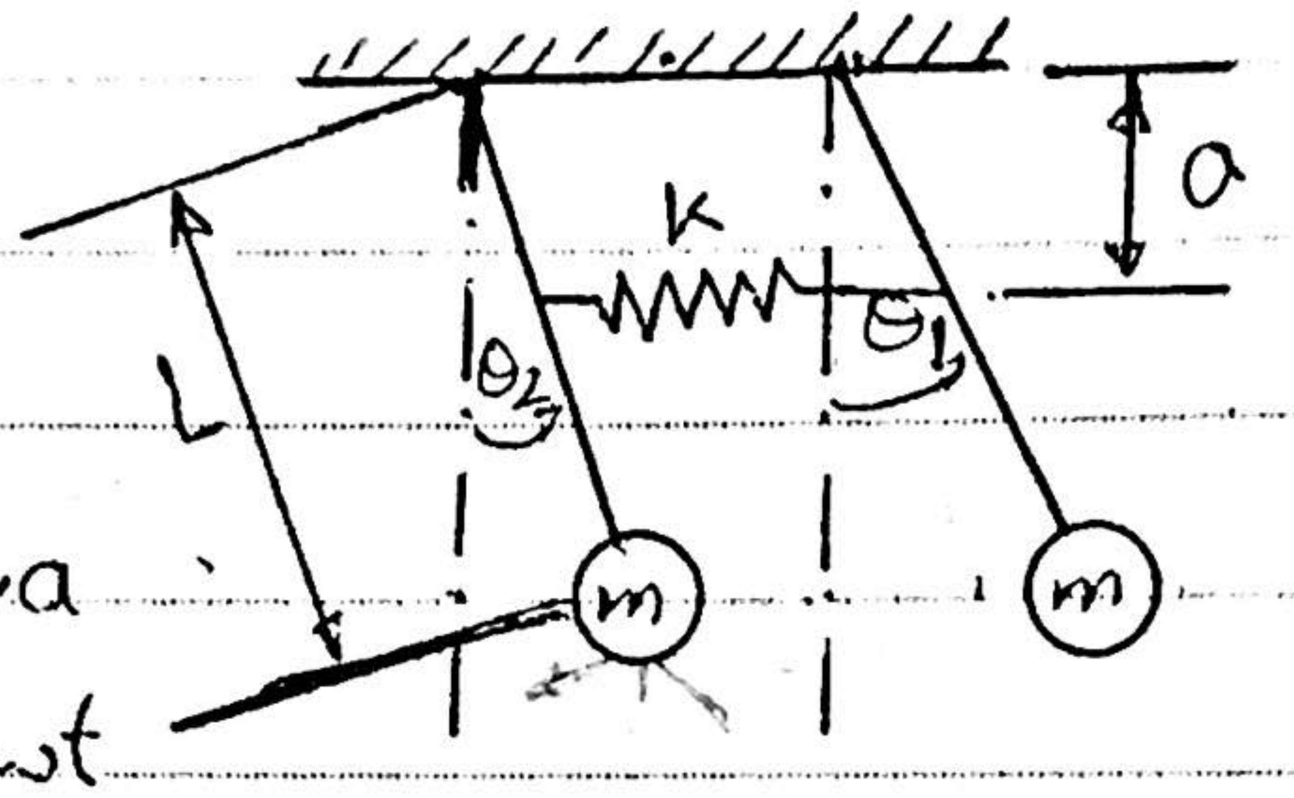




Example: Determin the equation of motion and the natural frequencies and mode shapes of the two degree of freedom system as shown below.

solution:

$\sum M = I\ddot{\theta}$



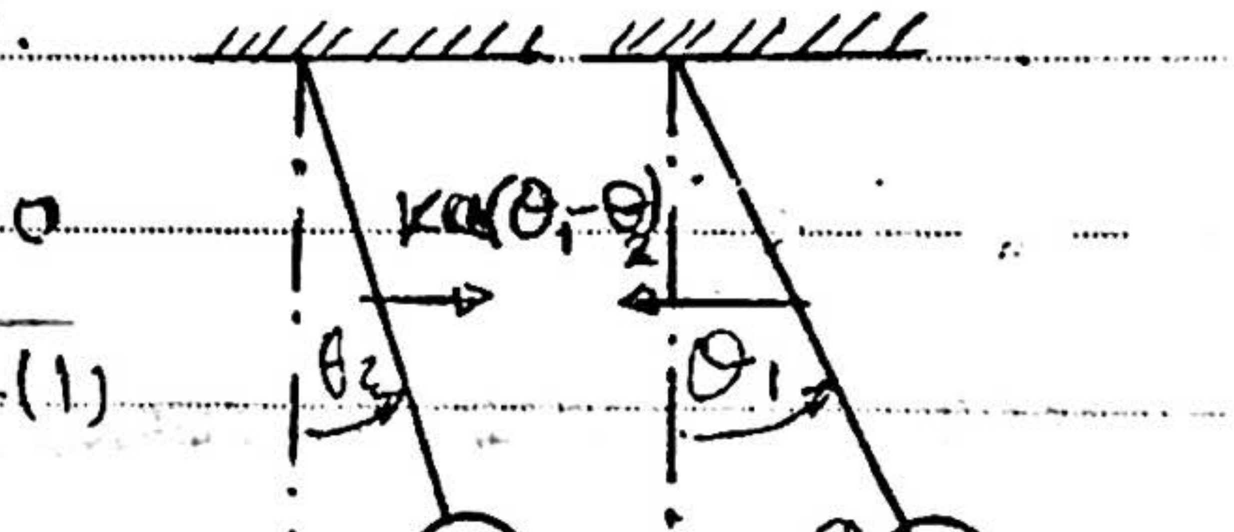
Let $I = mL^2$, $\sin\theta = \theta$

$\Rightarrow mL^2\ddot{\theta}_1 = -mgL\theta_1 - ka(\theta_1-\theta_2)\cdot a$

$mL^2\ddot{\theta}_2 = -mgL\theta_2 + ka(\theta_1-\theta_2)\cdot a$

let $\theta_1 = A_1\cos\omega t \Rightarrow \ddot{\theta}_1 = -A_1\omega^2\cos\omega t$

$\theta_2 = A_2\cos\omega t \Rightarrow \ddot{\theta}_2 = -A_2\omega^2\cos\omega t$



$\Rightarrow -mL^2\omega^2A_1 + mgLA_1 + ka^2A_1 - ka^2A_2 = 0$

$\Rightarrow (ka^2 + mgL - mL^2\omega^2)A_1 - ka^2A_2 = 0$ ---(1)

and $-mL^2\omega^2A + mgLA_2 + ka^2A_2 - ka^2A_1 = 0$

$\Rightarrow -ka^2A_1 + (ka^2 + mgL - mL^2\omega^2)A_2 = 0$ ---(2)

$$\Rightarrow \begin{vmatrix} ka^2+mgL-mL^2\omega^2 & -ka^2 \\ -ka^2 & ka^2+mgL-mL^2\omega^2 \end{vmatrix} = 0$$

$\Rightarrow K^2a^4 + ka^2mgL - ka^2mL^2\omega^2 + mgLka^2 + m^2g^2L^2 - m^2gL^3\omega^2 - mL^2\omega^2ka^2 - m^2gL^3\omega^2 + m^2L^4\omega^4 - \cancel{k^2a^4} = 0$

$\Rightarrow 2ka^2mgL - 2ka^2mL^2\omega^2 + m^2g^2L^2 - 2m^2gL^3\omega^2 + m^2L^4\omega^4 = 0$

$\Rightarrow m^2L^4\omega^4 - (2ka^2mL + 2m^2gL^3)\omega^2 + (2ka^2mgL + m^2g^2L^2) = 0$

let $\lambda = \omega^2$

$$\lambda^2 - \left(\frac{2ka^2mL^2}{m^2L^4} + \frac{2m^2gL^3}{m^2L^4}\right)\lambda + \left(\frac{2ka^2mgL}{m^2L^4} + \frac{m^2g^2L^2}{m^2L^4}\right) = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{\frac{2ka^2mL^2}{m^2L^4} + \frac{2m^2gL^3}{m^2L^4} \mp \sqrt{\left(\frac{2ka^2mL^2}{m^2L^4} + \frac{2m^2gL^3}{m^2L^4}\right)^2 - 4\left(\frac{2ka^2mgL}{m^2L^2} + \frac{m^2g^2L^2}{m^2L^4}\right)}}{2}$$

$$\lambda_{1,2} = \frac{\frac{2Ka^2}{mL^2}+\frac{2g}{L} \mp \sqrt{\left(\frac{4k^2a^4m^2L^4}{m^4L^8}+\frac{8m^3gL^5a^2k}{m^4L^8}+\frac{4m^4g^2L^6}{m^4L^8}\right) - \frac{8ka^2mgl}{m^2L^4} - \frac{4m^2g^2l^2}{m^2L^4}}}{2}$$

$$= \frac{ka^2}{mL^2}+\frac{g}{L} \mp \frac{1}{2}\sqrt{\frac{4k^2a^4}{m^2L^4}+\cancel{\frac{8ga^2k}{mL^3}}+\cancel{\frac{4g^2}{L^2}}-\cancel{\frac{8ka^2g}{mL^3}}-\cancel{\frac{4g^2}{L^2}}}$$

$$= \frac{ka^2}{mL^2}+\frac{g}{l} \mp \frac{ka^2}{mL^2}$$

$\therefore\ \lambda_1 = \frac{g}{L}$ and $\lambda_2 = \frac{g}{L} + 2\frac{ka^2}{mL^2}$

$\therefore\ \omega_1 = \sqrt{\frac{g}{L}}$, $\omega_2 = \sqrt{\frac{g}{L} + 2\frac{ka^2}{mL^2}}$

$\therefore$ From equations (1) and (2) we obtained to

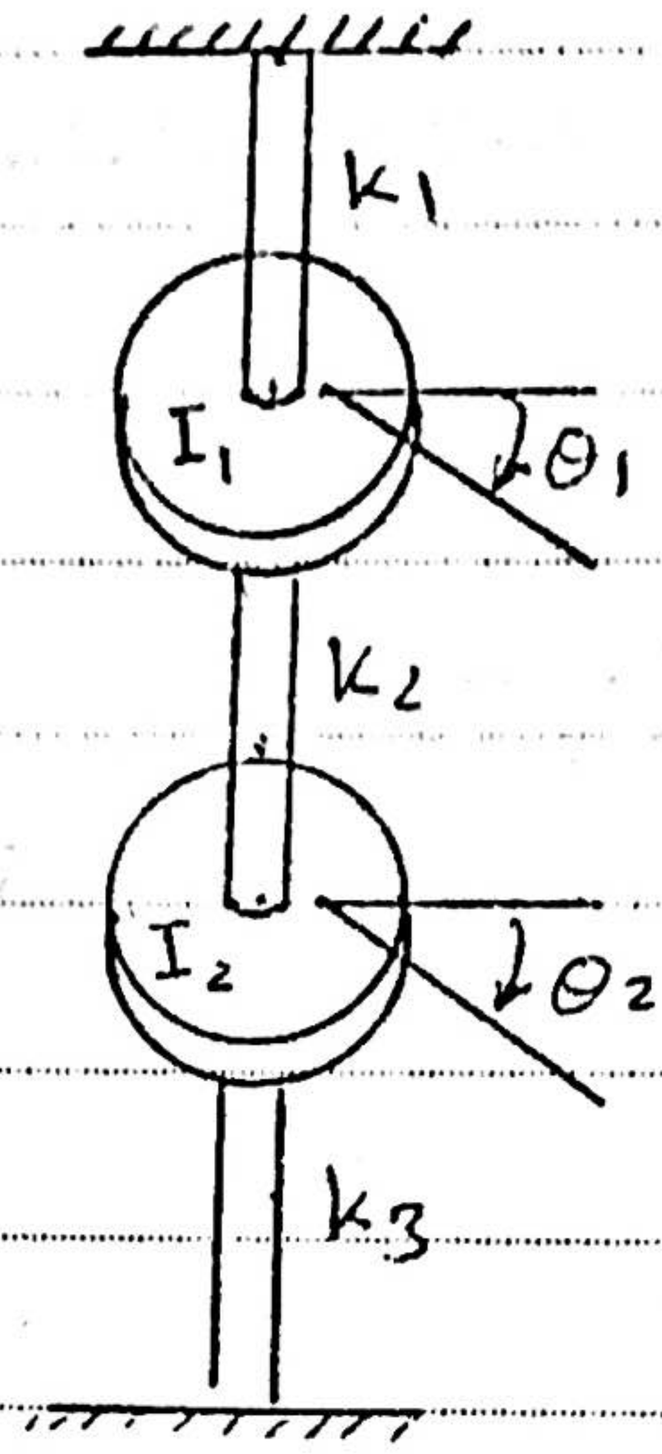
$\left(\frac{A_1}{A_2}\right)^{(1)} = 1.0$; $\left(\frac{A_1}{A_2}\right)^{(2)} = -1.0$

$\therefore\ \psi_1 = \begin{Bmatrix} 1.0 \\ 1.0 \end{Bmatrix}$; $\psi_2 = \begin{Bmatrix} -1.0 \\ 1.0 \end{Bmatrix}$

→ The normal mode oscillations are given by:

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}^{(1)} = A_1 \begin{Bmatrix} 1.0 \\ 1.0 \end{Bmatrix} \cos(\omega t + \phi_1) \text{ and } \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}^{(2)} = A_2 \begin{Bmatrix} -1.0 \\ 1.0 \end{Bmatrix} \cos(\omega t + \phi_2)$$

Example: Determine the differential equations for the torsional system with two degree of freedom as shown below:



solution:

For $\theta_2 > \theta_1$

$\Rightarrow$ $I_1 \ddot{\theta}_1 = -k_1\theta_1 + k_2(\theta_2 - \theta_1)$

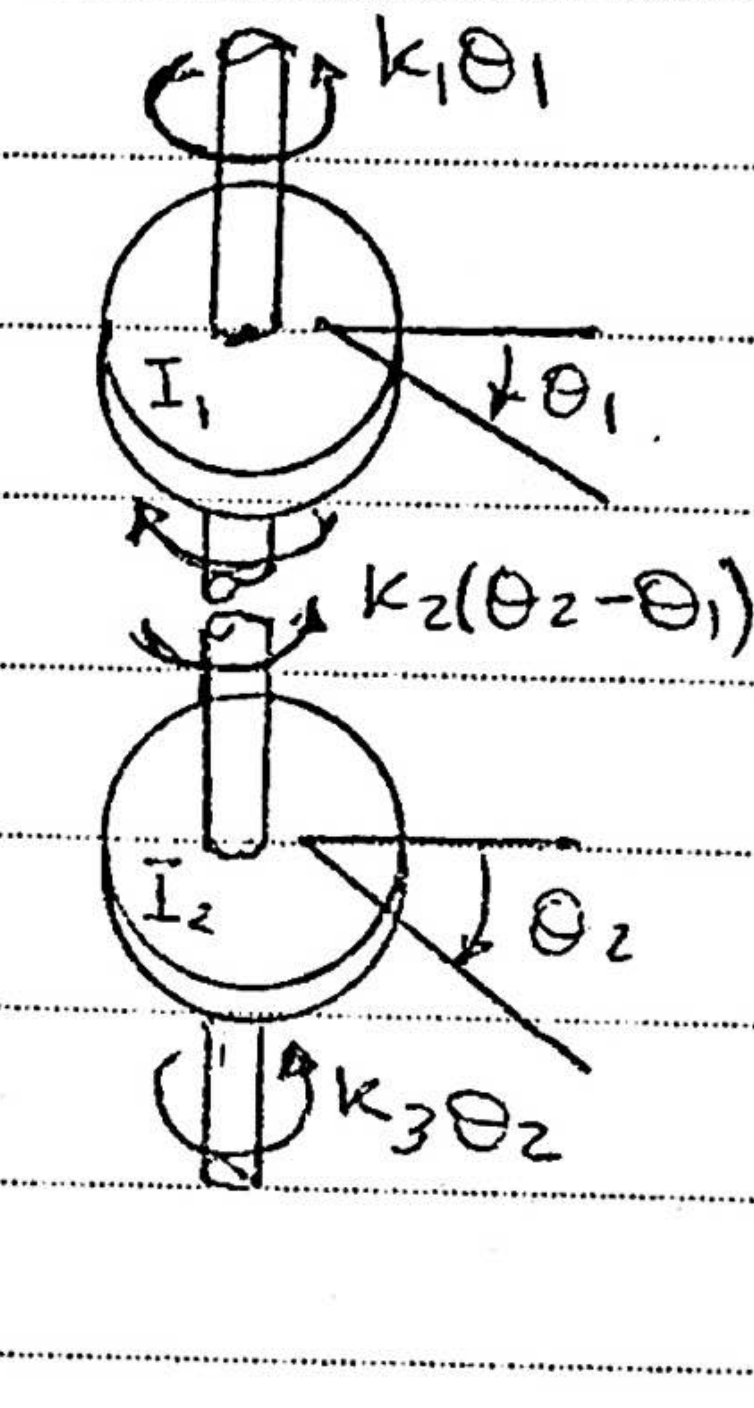
$I_2 \ddot{\theta}_2 = -k_2(\theta_2 - \theta_1) - k_3\theta_2$

$\Rightarrow$ $\ddot{\theta}_1 + \frac{k_1 + k_2}{I_1}\theta_1 - \frac{k_2}{I_1}\theta_2 = 0$

$\ddot{\theta}_2 + \frac{k_2 + k_3}{I_2}\theta_2 - \frac{k_2}{I_2}\theta_1 = 0$

Let $\theta_1 = \Theta_1 \sin\omega t$, $\ddot{\theta}_1 = -\Theta_1 \omega^2 \sin\omega t$

$\theta_2 = \Theta_2 \sin\omega t$; $\ddot{\theta}_2 = -\Theta_2 \omega^2 \sin\omega t$



$\Rightarrow$ $-\Theta_1\omega^2 + (k_1+k_2)/I_1\,\Theta_1 - \frac{k_2}{I_1}\Theta_2 = 0$

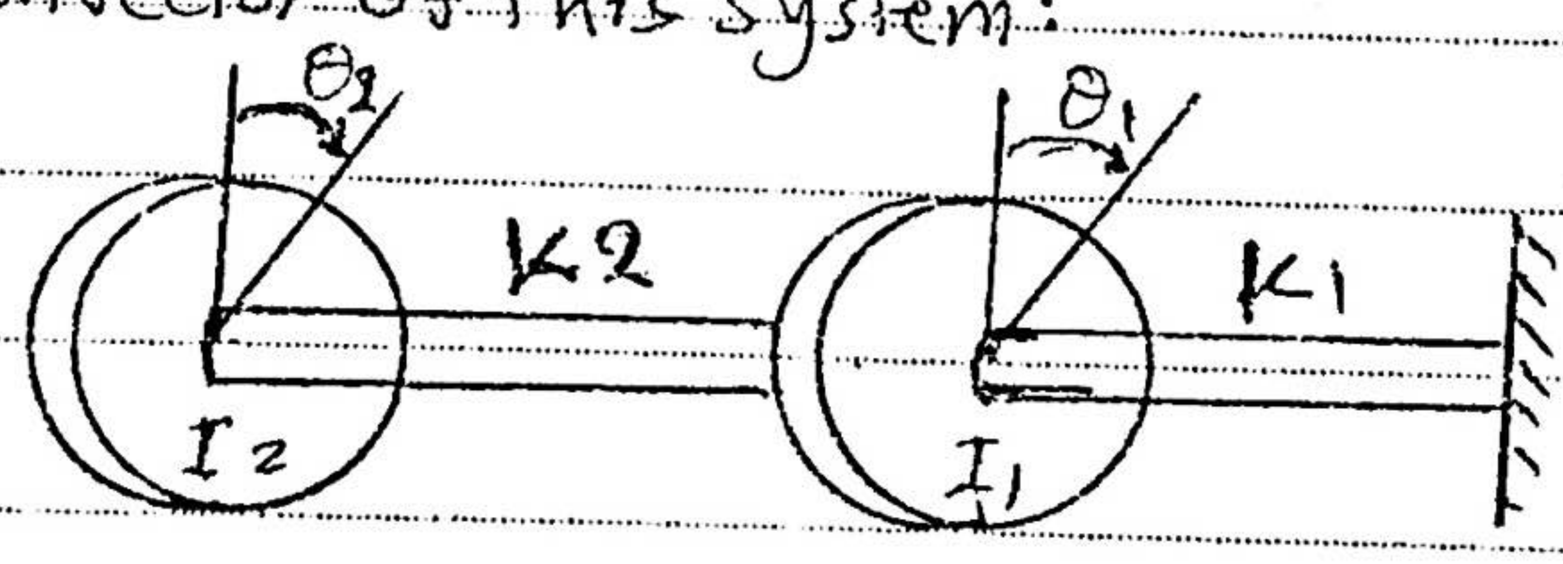
$-\Theta_2\omega^2 + (k_2+k_3)/I_2\,\Theta_2 - \frac{k_2}{I_2}\Theta_1 = 0$

$\Rightarrow$ $[(k_1+k_2)/I_1 - \omega^2]\Theta_1 - k_2\Theta_2 = 0$

$-k_2\Theta_1 + [(k_2+k_3)/I_2 - \omega^2]\Theta_2 = 0$

$$\Rightarrow \begin{vmatrix} (k_1+k_2)/I_1 - \omega^2 & -k_2/I_1 \\ -k_2/I_2 & (k_2+k_3)/I_2 - \omega^2 \end{vmatrix} = 0$$

H.w: Determine the eigenvector of this system:

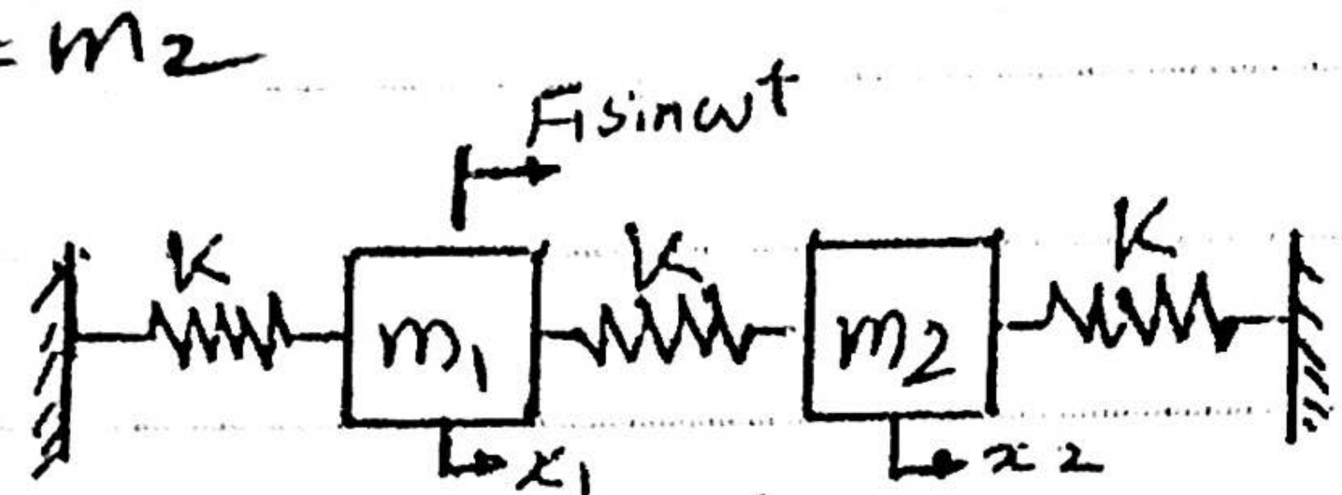


Two DoF system Effected by forces.

The system shown below when m is exsited by the force $F_1 \sin\omega t$. plot it's frequency response curve. Assume $m = m_1 = m_2$.

solution. by assuming $m_1 = m_2$

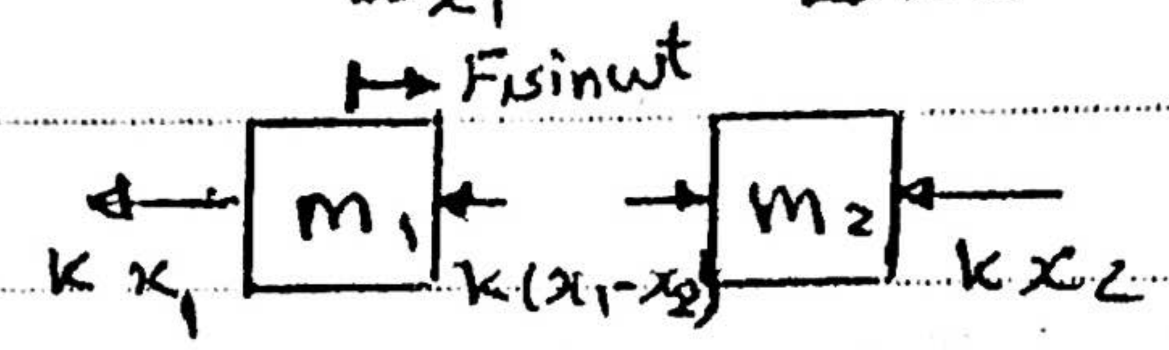
Σ equation of motion,



For(m_1): $m\vec{\ddot{x}}_1 = \vec{\Sigma F}$

$$m\ddot{x}_1 = F_1\sin\omega t - kx_1 - k(x_1 - x_2)$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = F_1\sin\omega t$$



For(m_2): $m\vec{\ddot{x}}_2 = \vec{\Sigma F} = k(x_1 - x_2) - kx_2$

$$m\ddot{x}_2 + 2kx_2 - kx_1 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin\omega t \quad -(1$$

Assume harmonic motion

$$\left(\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin\omega t * -\omega^2 \right) -(2$$

$$\begin{bmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \sin\omega t$$

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \quad \ldots *$$

$$\Rightarrow [Z(\omega)] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [Z(\omega)]^{-1} \begin{bmatrix} F_1 \\ 0 \end{bmatrix} = \frac{\operatorname{adj}[Z(\omega)]}{|Z(\omega)|} \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

$$\therefore [Z(\omega)] = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

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ملاحظة / لإيجاد سعة الذبذبة لمنظومة مثارة بقوة توافقية ($F \sin wt$,
$F \cos wt$) نتبع الخطوات التالية :

① $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \dfrac{adj\,[Z(w)] \cdot \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}}{|Z(w)|}$

① $[Z(w)] = \begin{vmatrix} 2K - mw^2 & -K \\ -K & 2K - mw^2 \end{vmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$ مصفوفة

② determin

$|Z(w)| = (2K - mw^2)^2 - K^2 = ((2K - mw^2) - K)((2K - mw^2) + K)$

$|Z(w)| = (K - mw^2)(3K - mw^2) = m^2(\frac{K}{m} - w^2)(\frac{3K}{m} - w^2)$

$\therefore |Z(w)| = m^2(w_1^2 - w^2)(w_2^2 - w^2)$

③ $adj[Z(w)] = \begin{bmatrix} 2K - mw^2 & K \\ K & 2K - mw^2 \end{bmatrix}$

adjoint

محاذن معكوس

④ $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \dfrac{\begin{vmatrix} 2K - mw^2 & K \\ K & 2K - mw^2 \end{vmatrix} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}}{m^2(w_1^2 - w^2)(w_2^2 - w^2)}$

⑤ $\therefore$ $X_1 = \dfrac{\begin{vmatrix} F_1 & K \\ 0 & 2K - mw^2 \end{vmatrix}}{m^2(w_1^2 - w^2)(w_2^2 - w^2)} \Rightarrow X_1 = \dfrac{(2K - mw^2) \cdot F_1}{m^2(w_1^2 - w^2)(w_2^2 - w^2)}$

$X_2 = \dfrac{\begin{vmatrix} 2K - mw^2 & F_1 \\ K & 0 \end{vmatrix}}{m^2(w_1^2 - w^2)(w_2^2 - w^2)} \Rightarrow X_2 = \dfrac{K \cdot F_1}{m^2(w_1^2 - w^2)(w_2^2 - w^2)}$

-- ANS

OR, by another method

From equation(*); put $F_1 = 0$ to find $\omega_{1,2}$

$$\Rightarrow \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0$$

Assume $\lambda = \omega^2$

$$\Rightarrow \begin{vmatrix} 2k - m\lambda & -k \\ -k & 2k - m\lambda \end{vmatrix} = 0$$

$$\Rightarrow 4k^2 - 4m\lambda k + m^2\lambda^2 - k^2 = 0 \Rightarrow \lambda^2 - 4\left(\frac{k}{m}\right)\lambda + 3\left(\frac{k}{m}\right)^2 = 0$$

$$\therefore \lambda_{1,2} = \frac{4\left(\frac{k}{m}\right) \pm \sqrt{16\left(\frac{k}{m}\right)^2 - 12\left(\frac{k}{m}\right)^2}}{2} \Rightarrow \lambda_{1,2} = (2 \pm 1)\frac{k}{m}$$

$$\Rightarrow \therefore \lambda_1 = \frac{k}{m} = \omega_1^2 \quad ; \text{ and } \quad \lambda_2 = \frac{3k}{m} = \omega_2^2$$

From equation(*) obtained to;

$$(2k - m\omega^2)\bar{X}_1 - k\bar{X}_2 = F_1 \quad \dots (1)$$

$$(2k - m\omega^2)\bar{X}_2 - k\bar{X}_1 = 0 \quad \dots (2)$$

From equation (2)

$$(2k - m\omega^2)\bar{X}_2 = k\bar{X}_1 \Rightarrow \bar{X}_2 = \frac{k\bar{X}_1}{(2k - m\omega^2)} \quad \dots (3)$$

Sub. in (1)

$$(2k - m\omega^2)\bar{X}_1 - \frac{k^2\bar{X}_1}{(2k - m\omega^2)} = F_1$$

$$\frac{(2k - m\omega^2)^2\bar{X}_1 - k^2\bar{X}_1}{(2k - m\omega^2)} = F_1$$

$$\left[(2k - m\omega^2)^2 - k^2\right]\bar{X}_1 = F_1(2k - m\omega^2)$$

$$\Rightarrow X_1 = \frac{(2k - m\omega^2)F_1}{(2k - m\omega^2 - k)(2k - m\omega^2 + k)}$$

$$= \frac{mF_1\left(2\frac{k}{m} - \omega^2\right)}{m^2\left(\frac{k}{m} - \omega^2\right)\left(\frac{3k}{m} - \omega^2\right)} = \frac{mF_1(2\omega_2 - \omega^2)}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

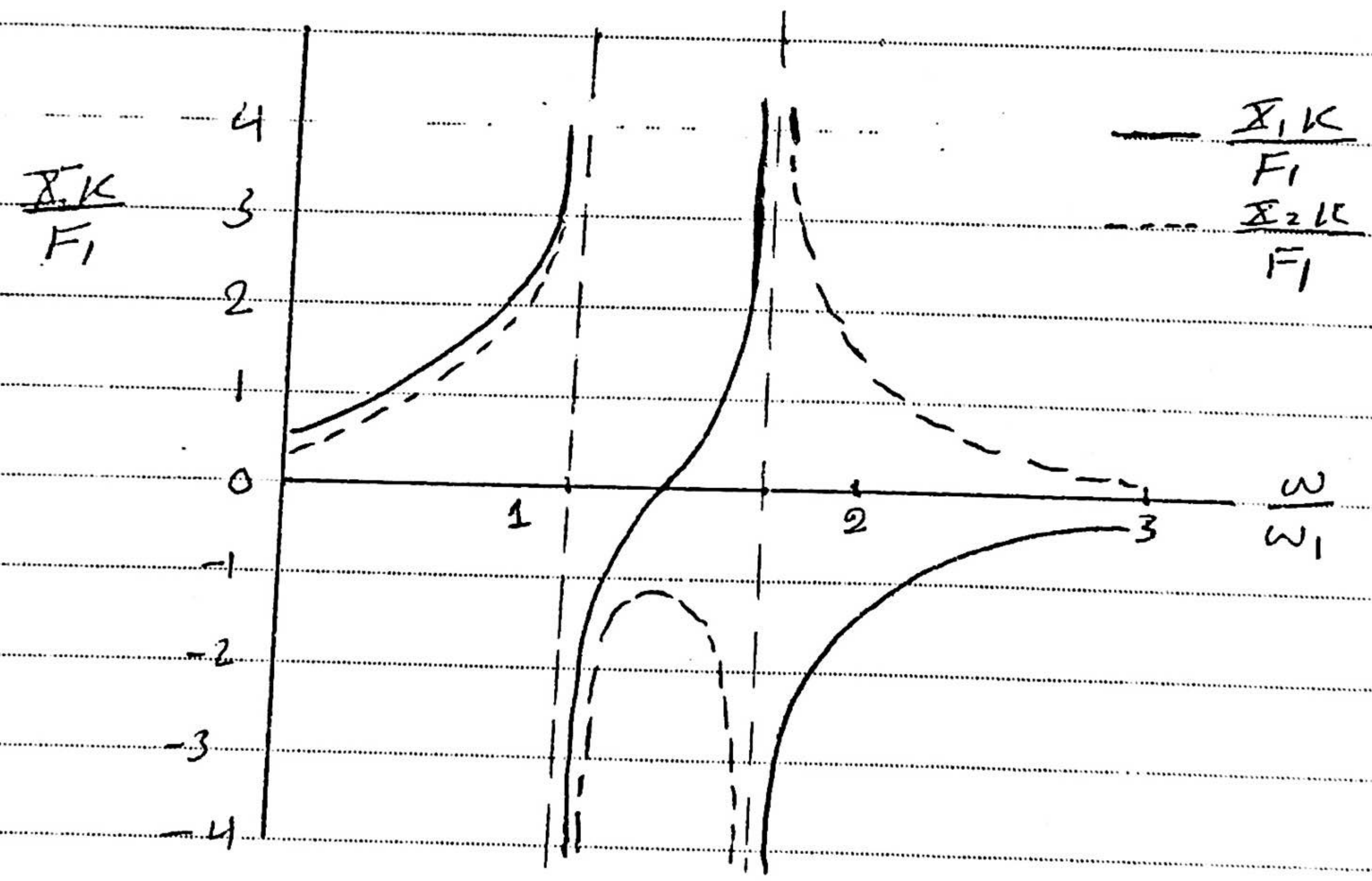
becouse m obtained from equation (2) $\Rightarrow m = m_2$

sub X_1 in equation (3)

$$\therefore X_2 = \frac{k}{(\cancel{2k - m\omega^2})} * \frac{(\cancel{2k - m\omega^2})F_1}{(2k - m\omega^2 - k)(2k - m\omega^2 + k)}$$

$$= \frac{kF_1}{(2k - m\omega^2 - k)(2k - m\omega^2 + k)}$$

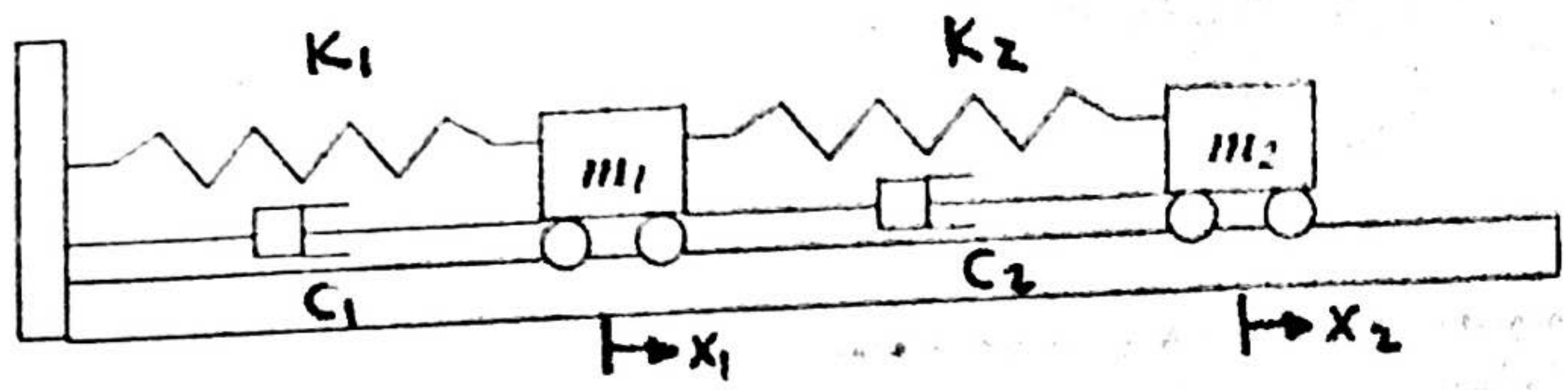
$$= \frac{kF_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$



H.W: If replace the force by $F_{10}\cos\omega t$ for the same system. Find the response.

TWO D.O.F With Damping

Example: Consider the following spring and mass system:



Undamped

For a system where the damping is negligible, we can apply Newton's second law to a free body diagram of the blocks and get the following for mass 1:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0,$$

and this for mass 2:

$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

where k_1 and k_2 are the spring constants, m_1 and m_2 are the masses of the blocks, and x_1 and x_2 are the displacements from equilibrium of each block.

Damped

The above is a special case of the more general case of a damped system. For a damped system, the free body diagrams and Netwon's second law yield:

for mass 1:

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = 0,$$

and this for mass 2:

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

This equation assumes that the vibrations are damped by the dashpots shown between the blocks and the wall. No external force is acting on the system. This system of two second order ordinary differential equations can be written in matrix form as:

$$M\ddot{x} + C\dot{x} + Kx = 0$$

where M represents the mass matrix:

$$M=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

and C represents the damping matrix

$$C=\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

and K represents the stiffness matrix

$$K=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

The acceleration, velocity, and displacement are given by the following three matrices, respectively:

$$\ddot{x}=\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} \quad \dot{x}=\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \quad x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The solution to this system of ODEs has the general form:

$$x(t)=Xe^{st}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{st}$$

For the case under consideration in the above simulation, which is the underdamped case, the solution is given by the following two equations:

$$x_1(t)=A_{1.1}e^{-\xi_1\omega_1 t}\cos(\omega_{1d}t)+A_{1.2}e^{-\xi_2\omega_2 t}\cos(\omega_{2d}t)$$

and

$$x_2(t)=A_{2.1}e^{-\xi_1\omega_1 t}\cos(\omega_{1d}t)+A_{2.2}e^{-\xi_2\omega_2 t}\cos(\omega_{2d}t)$$

where the damping factor is defined as

$$\xi=c/C_c$$

and the damped natural frequency is defined as

$$\omega_d = \omega\sqrt{1-\xi^2}$$

Note that ω_1 and ω_2 represent the first and second natural frequencies of the system and are related to the two modal vibrations that combine to provide the overall motion of the system. They are not related to the ratio of a single spring constant and mass as in a single degree of freedom system. This means that

$$\omega_1 \neq \sqrt{\frac{k_1}{m_1}} \quad \text{and} \quad \omega_2 \neq \sqrt{\frac{k_2}{m_2}}$$

The natural frequencies, in rad /sec, of the two degree of freedom system, ω_1 and ω_2, are proportional the eigenvalues of the stiffness and mass matrices and are given by:

$$\omega_1 = \sqrt{\lambda(1,1)} \quad \text{and} \quad \omega_2 = \sqrt{\lambda(2,2)}$$

where λ represents the eigenvalue function. Recall that the eigenvalue is defined such that:

$$\left|\vec{K} - \lambda\vec{M}\right| = 0$$

Solving for the eigenvalues will thus provide us with the natural frequencies of the system. In order to find the relative amplitudes of the masses vibrations, it is also necessary to find the eigenvector matrices. Note that by setting $A_{1,1} = 1$ and $A_{2,1} = 1$, future calculations can be simplified while still maintaining the correct calculations—the concern here is with the relative amplitude of the vibration.

Recall that an eigenvector A is defined such that:

$$\left[\vec{K} - \lambda\vec{M}\right] \bullet \left[A\right] = 0$$

The eigenvectors are given by

$$\vec{A}_1 = \begin{bmatrix} 1 \\ A_{1,2} \end{bmatrix} \quad \text{and} \quad \vec{A}_2 = \begin{bmatrix} 1 \\ A_{2,2} \end{bmatrix}$$

For a two degree of freedom system, given the mass and stiffness matrices above, the eigenvector calculations are fairly simple:

$$A_{i,2} = \frac{(k_1 + k_2) - \omega_i^2 m_1}{k_2}$$

At this point, then, define two functions $p_1(t)$ and $p_2(t)$, and a composite matrix Φ as follows:

$$\vec{\Phi} = \begin{bmatrix} 1 & 1 \\ A_{1,2} & A_{2,2} \end{bmatrix}$$

$$p_1(t) = p_{10} e^{-\xi_1 \omega_1 t} \cos(\omega_{1d} t)$$
$$p_2(t) = p_{20} e^{-\xi_2 \omega_2 t} \cos(\omega_{2d} t)$$

$p_1(t)$ represents the first modal motion of the system.

$p_2(t)$ represents the second modal motion of the system

From our earlier equations for equations for x, it is clear that:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \vec{\Phi} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

The initial conditions, then will provide us with p10 and p20, which are the last pieces of the displacement function. Calculating the inverse of F and multiplying it by the column matrix x10, x20 will give us those values. Then the complete displacement x as a function of time t is given by:

$$x_1(t) = p_1(t) + p_2(t)$$
$$x_2(t) = A_{1,2} p_1(t) + A_{2,2} p_2(t)$$

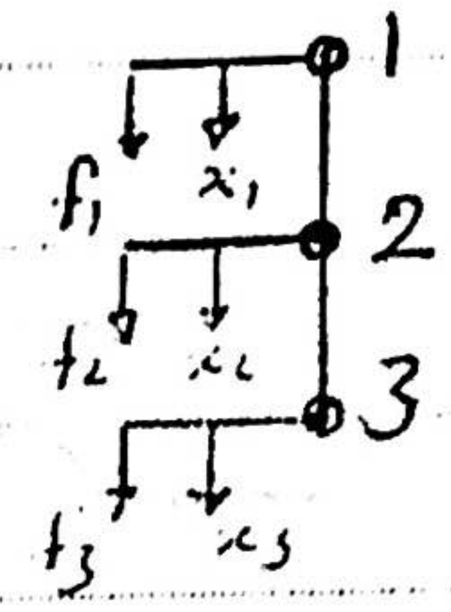
Chapter Six

Multi Degree of Freedom

◎ Flexibility and Stiffness Matrix:

Flexibility influence coefficient (a_{ij})

is the displacement at (i) due to a unit force at (j).



$$\left.\begin{aligned} x_1 &= a_{11} f_1 + a_{12} f_2 + a_{13} f_3 \\ x_2 &= a_{21} f_1 + a_{22} f_2 + a_{23} f_3 \\ x_3 &= a_{31} f_1 + a_{32} f_2 + a_{33} f_3 \end{aligned}\right\} \Rightarrow \{x\} = [a]\{f\}$$

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \; ; \; \{f\} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \quad \text{and;}$$

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ ..... flexibility matrix.}$$

$$\{x\} = [a]\{f\} \Rightarrow [a]^{-1}\{x\} = \{f\} \Rightarrow [k]\{x\} = \{f\}$$

where; $[k] = [a]^{-1}$ or; $[a] = [k]^{-1}$

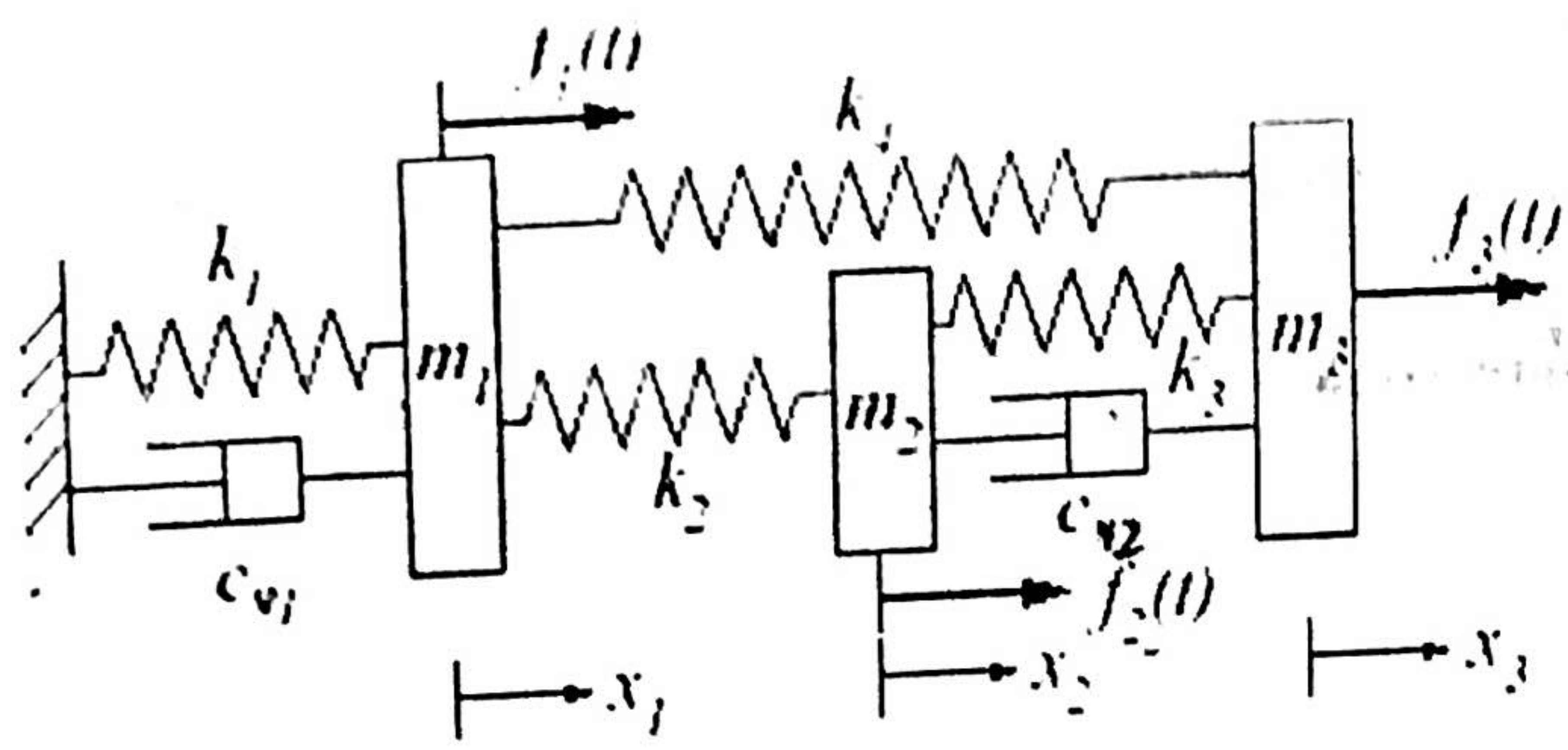
$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \text{ ..... stiffness matrix.}$$

* Reciprocity Theorem, $a_{ij} = a_{ji}$

$k_{ij} = k_{ji}$

MODAL DAMPING :-

Example : Consider the 3 degree-of-freedom system,

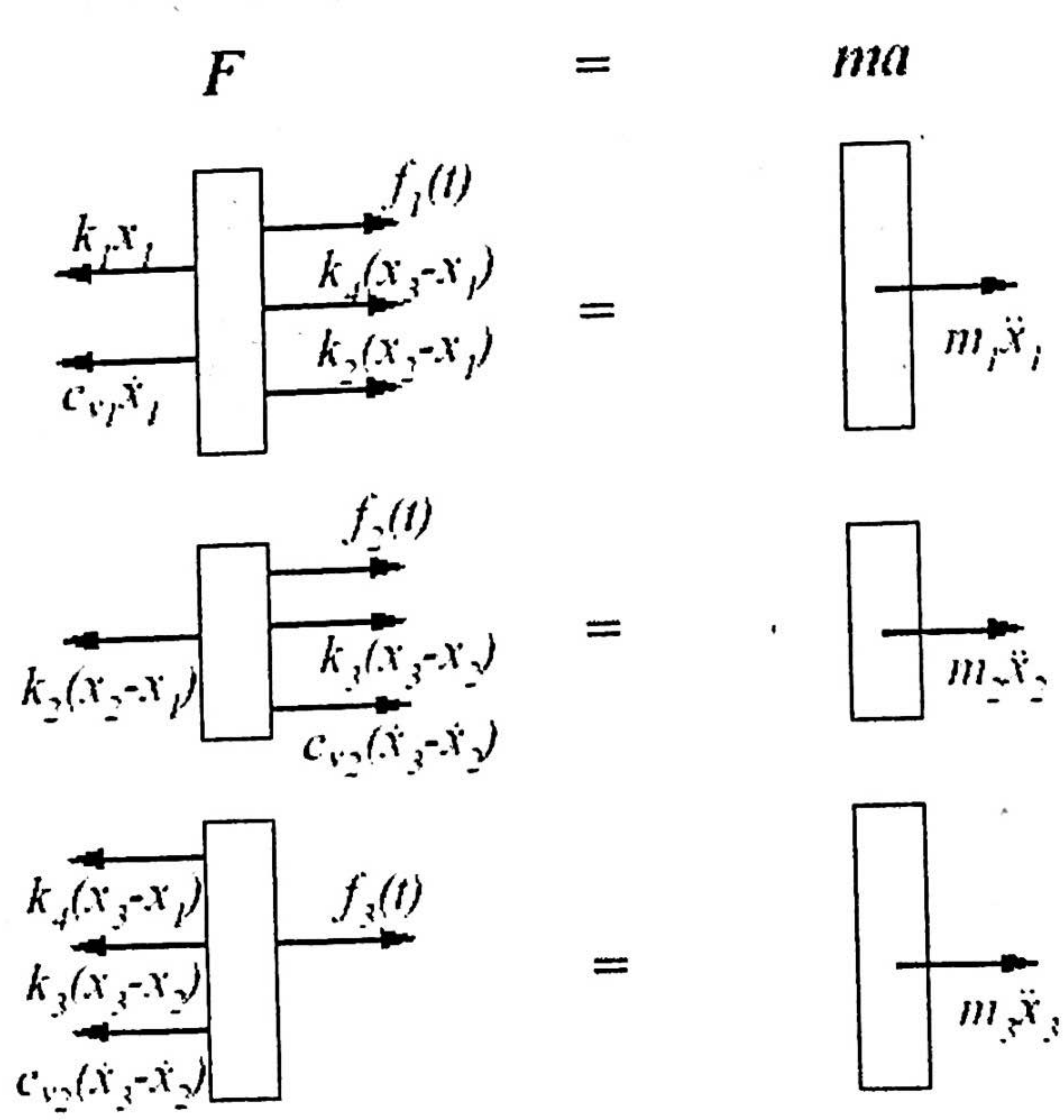


Solution:

There are 3 degrees of freedom in this problem since to fully characterize the system we must know the positions of the three masses (x_1, x_2, and x3).

Three free body diagrams are needed to form the equations of motion. However, it is also possible to form the coefficient matrices directly, since each parameter in a mass-dashpot-spring system has a very distinguishable role.

The equations of motion can be obtained from free body diagrams, based on the Newton's second law of motion, F = m*a.



The equations of motion can therefore be expressed as,

$$m_1\ddot{x}_1 + c_{v1}\dot{x}_1 + (k_1 + k_2 + k_4)x_1 - k_2 x_2 - k_4 x_3 = f_1(t)$$
$$m_2\ddot{x}_2 + c_{v2}\dot{x}_2 - c_{v2}\dot{x}_3 + (k_2 + k_3)x_2 - k_2 x_1 - k_3 x_3 = f_2(t)$$
$$m_3\ddot{x}_3 + c_{v2}\dot{x}_3 - c_{v2}\dot{x}_2 + (k_3 + k_4)x_3 - k_3 x_2 - k_4 x_1 = f_1(t)$$

In matrix form the equations become,

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c_{v1} & 0 & 0 \\ 0 & c_{v2} & -c_{v2} \\ 0 & -c_{v2} & c_{v2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} +$$
$$\begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_2 + k_3 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

Note/ If Free vibration $f_1(t) = 0$ $f_2(t) = 0$ $f_3(t) = 0$